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Conifold type singularities, $\mathcal{N}=2$ Liouville and $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ theories

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Abstract: In this paper we discuss various aspects of non-compact models of CFT of the type: $\prod_{j=1}^{N_L}\{\mathcal{N}=2 \text{ Liouville theory}\}_j \otimes \prod_{i=1}^{N_M}\{\mathcal{N}=2 \text{ minimal model}\}_i$ and $\prod_{j=1}^{N_L}\{\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1) \text{ supercoset}\}_j \otimes \prod_{i=1}^{N_M}\{\mathcal{N}=2 \text{ minimal model}\}_i$. These models are related to each other by T-duality. Such string vacua are expected to describe non-compact Calabi-Yau compactifications, typically ALE fibrations over (weighted) projective spaces. We find that when the Liouville ($\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$) theory is coupled to minimal models, there exist only (c,c), (a,a) ((c,a), (a,c))-type of massless states in CY 3 and 4-folds and the theory possesses only complex (Kähler) structure deformations. Thus the space-time has the characteristic feature of a conifold type singularity whose deformation (resolution) is given by the $\mathcal{N}=2$ Liouville ($\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$) theory.
Spectra of compact BPS D-branes determined from the open string sector are compared with those of massless moduli. We compute the open string Witten index and determine intersection numbers of vanishing cycles. We also study non-BPS branes of the theory that are natural extensions of the "unstable B-branes" of the SU(2) WZW model in [31].

Keywords: D-branes, Superstring Vacua, Conformal Field Models in String Theory.

Contents

1. Introduction

Superstring theories on non-compact curved backgrounds are receiving a great deal of attentions. Well-defined description of these string vacua by irrational superconformal field theories (SCFT's) is an important and challenging problem. Recently, considerable progress has been made in the study of the $\mathcal{N}=2$ Liouville theory or the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki supercoset theory, based on the method of modular bootstrap and the exact

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description of D-branes in terms of boundary states [1]–[11]. It should be also mentioned that attempts of the conformal (boundary) bootstrap in these $\mathcal{N}=2$ systems are given in [2, 6, 7, 10].

In this paper we investigate the superstring vacua of the type

$$\prod_{j=1}^{N_L}\{\mathcal{N}=2 \text{ Liouville}\}_j \otimes \prod_{i=1}^{N_M}\{\mathcal{N}=2 \text{ minimal model}\}_i$$

which in general contain more than one $\mathcal{N}=2$ Liouville fields coupled to $\mathcal{N}=2$ minimal model. A suitable orbifolding procedure is imposed as in the Gepner construction [12] in order to ensure the space-time SUSY. If one uses the T-duality (mirror symmetry) of $\mathcal{N}=2$ Liouville to $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model [13, 14, 15], the above models are equivalent to

$$\prod_{j=1}^{N_L}\{\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1) \text{ supercoset}\}_j \otimes \prod_{i=1}^{N_M}\{\mathcal{N}=2 \text{ minimal model}\}_i\,.$$

These models are expected to describe non-compact Calabi-Yau manifolds where we obtain non-gravitational space-time theories due to the Liouville mass gap. The earlier studies on such models are given in e.g. [14] and [16]–[27]. Many related topics and a detailed list of literature are found in a review paper [28].

A basic idea in Gepner construction is the orbifolding with respect to the $\mathrm{U}(1)_R$-charge of $\mathcal{N}=2$ superconformal algebra (SCA). One way to impose charge-integrality is to consider spectral-flow orbits as in [29]: by using flow-invariant orbits we can systematically construct conformal blocks of the theory. In our previous paper [1], we have considered $\mathcal{N}=2$ Liouville theory with rational central charges and introduced extended characters which are defined by an infinite sum over spectral flows of irreducible $\mathcal{N}=2$ characters. We have shown that

- Extended characters have discrete and finite spectra of $\mathrm{U}(1)_R$-charges, although they may have continuous spectra of conformal weights.
- They are closed under modular transformations.

We have also noticed that these characters naturally appear in the torus partition functions of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki models [4] (see also [5]), which are T-dual to the $\mathcal{N}=2$ Liouville theories. In this paper we use extended characters together with irreducible characters of minimal models as basic building blocks of our construction.

In the following we especially study models with $N_M=1$ and $1\leq N_L\leq 3$, which are interpreted geometrically as ALE fibrations over (weighted) projective spaces [23, 26]. We find a very interesting aspect of massless spectrum in these models: in the case of $\mathcal{N}=2$ Liouville theory coupled to minimal models there exist only (c,c) or (a,a)-type massless states in the CY 3 and 4-folds and no (c,a) or (a,c) massless states appear. Thus the theory possesses only complex structure deformations and no deformations of Kähler structure. On the other hand, if we use the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ description, the theory possesses only (c,a)

and (a,c)-type massless states and the moduli of Kähler structure deformations. Thus the space-time has the characteristic feature of a conifold type singularity which is deformed (resolved) by the $\mathcal{N}=2$ Liouville $(\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1))$ theory.

In the case of models describing non-compact K3 surfaces or smoothed ADE type singularity, on the other hand, the same number of $(a,c),(c,a)$ and $(c,c),(a,a)$ states appear in accord with the $\mathcal{N}=4$ world-sheet supersymmetry.

This paper is organized as follows: in section 2 we present a brief review on the irreducible and extended characters in the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki models following [4]. In section 3 we study the closed string sector of our non-compact models. We analyze the torus partition functions, elliptic genera and the massless spectra of closed string states. We study models with $N_M=1$ and $1\leq N_L\leq 3$, and find the interesting characteristics of their massless spectra as mentioned above.

In section 4, we study the open string sector of our models. We focus on the BPS compact branes and evaluate the cylinder amplitudes. We compare the spectra of BPS compact branes with those of massless moduli determined in section 3. We find that some of the BPS branes (cycles) are not associated with massless moduli as noticed previously in the case of singular CY manifolds [30, 20, 21]. We also derive the general formula of open string Witten indices and prove the conjecture of [23]. We further construct boundary states for a class of non-BPS D-branes which are extensions of “unstable B-branes” in the SU(2)-WZW model [31]. Contrary to the flat case, non-BPS branes including RR-components (but with vanishing RR-charges) also exist. This type of branes could be identified with the ones studied recently in [32] using the DBI action. Section 5 is devoted to a summary and discussions. We present in appendix E some consistency checks of our modular transformation formulas with the known results about the higher level Appell functions [33, 34].

In the following we mainly use the language of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ supercoset theory rather than the $\mathcal{N}=2$ Liouville theory for the sake of convenience. However, later in section 3 we identify results of CFT analyses as describing the deformed geometries based on $\mathcal{N}=2$ Liouville theory.

2. Preliminaries

We start with a brief review on the conformal blocks and their modular properties of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model. More complete arguments are given in [4] (see also [5]).

2.1 Branching functions in $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model

The Kazama-Suzuki model for $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ is defined as the coset CFT

$$\frac{\mathrm{SL}(2;\mathbb{R})_\kappa\times \mathrm{SO}(2)_1}{\mathrm{U}(1)_{-(\kappa-2)}}, \tag{2.1}$$

which is an $\mathcal{N}=2$ SCFT with $\hat{c}(\equiv c/3)=1+2/k$, $(k\equiv\kappa-2)$. The coset characters are defined by the following branching relation

$$\chi_\xi\left(\tau,\frac{2}{k}z+w\right)\frac{\theta_3\left(\tau,\frac{k+2}{k}z+w\right)}{\eta(\tau)}=\sum_m\chi^{(\mathrm{NS})}_{\xi,m}(\tau,z)\frac{q^{-\frac{m^2}{k}}e^{2\pi imw}}{\eta(\tau)}\,, \tag{2.2}$$

where ξ labels irreducible representations of $\widehat{\mathrm{SL}}(2;\mathbb{R})_\kappa$ and $\chi_\xi(\tau,u)$ denotes its character. We can identify the branching functions $\chi^{(\mathrm{NS})}_{\xi,m}(\tau,z)$ with the irreducible characters of $\mathcal{N}=2$ SCA as follows:

- **for the continuous series $\xi=\hat{\mathcal{C}}_{p,m}$:**

$$\chi_{\mathbf{c}}{}^{(\mathrm{NS})}_{p,m}(\tau,z)=q^{\frac{p^2}{2}+\frac{m^2}{k}}\,e^{2\pi i\frac{2m}{k}z}\,\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \tag{2.3}$$

which are the massive characters of $\mathcal{N}=2$ SCA. The highest-weight state has the quantum numbers, conformal dimension and U(1)-charge as

$$h=\frac{p^2}{2}+\frac{1}{4k}+\frac{m^2}{k}\,,\qquad Q=\frac{2m}{k}\,. \tag{2.4}$$

- **for the discrete series $\xi=\hat{\mathcal{D}}^+_j$:**

$$\chi_{\mathbf{d}}{}^{(\mathrm{NS})}_{j,m=j+n}(\tau,z)=\frac{q^{\frac{j+n^2+2nj}{k}-\frac{1}{4k}}e^{2\pi i\frac{2(j+n)}{k}z}}{1+e^{2\pi iz}q^{n+1/2}}\,\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,,\qquad(^\forall n\in\mathbb{Z})\,, \tag{2.5}$$

which are the n-step spectral flow of massless matter characters. The n-step flow is generated by an operator $U_n\equiv e^{in\Phi}$ where Φ denotes the zero mode of a scalar field of the $\mathcal{N}=2$ U(1) current $J\equiv i\partial\Phi$. The unitarity requires the condition [35, 36]

$$0<j<\frac{\kappa}{2}\left(\equiv\frac{k+2}{2}\right). \tag{2.6}$$

The highest-weight states have the following quantum numbers;

$$h=\frac{2j\left(n+\frac{1}{2}\right)+n^2}{k}\,,\qquad Q=\frac{2(j+n)}{k}\,,\qquad(n\geq 0)\,, \tag{2.7}$$

$$h=\frac{-\left(k-2j\right)\left(n+\frac{1}{2}\right)+n^2}{k}\,,\qquad Q=\frac{2(j+n)}{k}-1\,,\qquad(n<0)\,. \tag{2.8}$$

They are given explicitly by $(j_0^+)^n|j,j\rangle\otimes|0\rangle_\psi$ $(n\geq 0)$, $(j^-_{-1})^{|n|-1}|j,j\rangle\otimes\psi^-_{-1/2}|0\rangle_\psi$ $(n<0)$ respectively (here $|j,j\rangle$ denotes the lowest weight state of bosonic $\mathrm{SL}(2;\mathbb{R})$ algebra and $|0\rangle_\psi$ denotes the fermion Fock vacuum). The highest weight representations $\hat{\mathcal{D}}^-_j$ merely yield the same type of characters and we need not take account of them.

- **for the identity representation $\xi=\mathbf{id}$:**

$$\chi^{(\mathrm{NS})}_{0,n}(\tau,z)=q^{-\frac{1}{4k}}\frac{(1-q)q^{\frac{n^2}{k}+n-\frac{1}{2}}e^{2\pi i\left(\frac{2n}{k}+1\right)z}}{\left(1+e^{2\pi iz}q^{n+1/2}\right)\left(1+e^{2\pi iz}q^{n-1/2}\right)}\,\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \tag{2.9}$$

which are the spectrally flowed graviton characters. The vacuum states are summarized as

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$n = 0$: the vacuum is $|0,0\rangle \otimes |0\rangle_\psi$ with $h = Q = 0$.

$n \geq 1$: the vacuum is $(j^+_{-1})^{n-1}|0,0\rangle \otimes \psi^+_{-1/2}|0\rangle_\psi$, which has the quantum numbers

$$h = \frac{n^2}{k} + n - \frac{1}{2}, \qquad Q = \frac{2n}{k} + 1\,. \tag{2.10}$$

$n \leq -1$: the vacuum is $(j^-_{-1})^{|n|-1}|0,0\rangle \otimes \psi^-_{-1/2}|0\rangle_\psi$, which has the quantum numbers

$$h = \frac{n^2}{k} - n - \frac{1}{2}, \qquad Q = \frac{2n}{k} - 1\,. \tag{2.11}$$

2.2 Extended Characters

From now on, we shall concentrate on models with a rational level $k = N/K$ ($N, K \in \mathbb{Z}_{>0}$). We define the extended characters by taking the mod N spectral flow sums of irreducible characters. In the following definitions, m is assumed to be an integral parameter within the range $-NK \leq m < NK$.

- **continuous representation ('extended massive character'):**

  we define

$$\chi_{\mathbf{c}}{}^{(\mathrm{NS})}(p, m; \tau, z) \equiv \sum_{n \in N\mathbf{Z}} \chi_{\mathbf{c}}{}^{(\mathrm{NS})}_{p, m/2K+n}(\tau, z) \equiv q^{\frac{p^2}{2}} \Theta_{m,NK}\left(\tau, \frac{2z}{N}\right) \frac{\theta_3(\tau, z)}{\eta(\tau)^3}\,, \tag{2.12}$$

  which has the highest-weight state with

$$h = \frac{p^2}{2} + \frac{m^2 + K^2}{4NK}\,, \qquad Q = \frac{m}{N}\,. \tag{2.13}$$

- **discrete representation ('extended massless matter character'):**

  we define (with the reparameterization $j \equiv s/(2K)$, $s \in \mathbb{Z}$)

$$\chi_{\mathbf{d}}{}^{(\mathrm{NS})}(s, m; \tau, z) \equiv \begin{cases} \sum_{n \in N\mathbf{Z}} \chi_{\mathbf{d}}{}^{(\mathrm{NS})}_{\frac{s}{2K}, \frac{m}{2K}+n}(\tau, z) & m \equiv s \pmod{2K} \\ 0 & m \not\equiv s \pmod{2K} \end{cases} \tag{2.14}$$

  where the unitarity condition (2.6) imposes

$$1 \leq s \leq N + 2K - 1\,, \qquad (s \in \mathbb{Z})\,. \tag{2.15}$$

  The vacuum vectors for $\chi_{\mathbf{d}}{}^{(\mathrm{NS})}(s, m = s + 2Kr)$, $(-N/2 \leq r < N/2)$ are characterized by

$$\begin{aligned} h &= \frac{Kr^2 + \left(r + \frac{1}{2}\right)s}{N}\,, \qquad Q = \frac{s + 2Kr}{N}\,, \qquad \left(0 \leq r < \frac{N}{2}\right) \\ h &= \frac{Kr^2 - \left(r + \frac{1}{2}\right)(N - s)}{N}\,, \qquad Q = \frac{s - N + 2Kr}{N}\,, \qquad \left(-\frac{N}{2} \leq r < 0\right)\,. \end{aligned} \tag{2.16}$$

- **identity representation ('extended graviton character'):**

  we define

$$\chi_0^{(\mathrm{NS})}(m;\tau,z) \equiv \begin{cases} \sum_{n\in N\mathbf{Z}} \chi^{(\mathrm{NS})}_{0,\frac{m}{2K}+n}(\tau,z) & m\in 2K\mathbb{Z} \\ 0 & m\notin 2K\mathbb{Z} \end{cases} \tag{2.17}$$

  The vacua for $\chi_0^{(\mathrm{NS})}(m=2Kr;\tau,z)$, $(-N/2\le r<N/2)$ are given as

$$\begin{aligned} &h = Q = 0\,, \qquad (r=0)\,,\\ &h=\frac{Kr^2}{N}+|r|-\frac{1}{2}\,, \qquad Q=\frac{2Kr}{N}+\mathrm{sgn}(r)\,, \qquad (r\neq 0)\,. \end{aligned} \tag{2.18}$$

The extended characters of other spin structures are defined by the 1/2-spectral flow;

$$\begin{aligned} \chi_*^{(\widetilde{\mathrm{NS}})}(*,m;\tau,z) &\equiv e^{-i\pi\frac{m}{N}}\,\chi_*^{(\mathrm{NS})}\left(*,m;\tau,z+\frac{1}{2}\right)\,,\\ \chi_*^{(\mathrm{R})}(*,m+K;\tau,z) &\equiv q^{\frac{\hat{c}}{8}}e^{i\pi\hat{c}z}\,\chi_*^{(\mathrm{NS})}\left(*,m;\tau,z+\frac{\tau}{2}\right)\,,\\ \chi_*^{(\widetilde{\mathrm{R}})}(*,m+K;\tau,z) &\equiv e^{-i\pi\frac{m}{N}}q^{\frac{\hat{c}}{8}}e^{i\pi\hat{c}z}\,\chi_*^{(\mathrm{NS})}\left(*,m;\tau,z+\frac{\tau}{2}+\frac{1}{2}\right)\,. \end{aligned} \tag{2.19}$$

Note that extended characters of discrete and identity representations in R and $\widetilde{\mathrm{R}}$-sectors take non-zero values only if $m\equiv s-K \pmod{2K}$, $m\in K(2\mathbb{Z}+1)$, respectively. The quantum numbers of the NS and R vacua are related by

$$\begin{aligned} h^{(\mathrm{R})}(*,m+K) &= h^{(\mathrm{NS})}(*,m+K)+\frac{1}{8}\equiv h^{(\mathrm{NS})}(*,m)+\frac{1}{2}Q^{(\mathrm{NS})}(m)+\frac{\hat{c}}{8}\,,\\ Q^{(\mathrm{R})}(m+K) &= Q^{(\mathrm{NS})}(m+K)+\frac{1}{2}\equiv Q^{(\mathrm{NS})}(m)+\frac{\hat{c}}{2}\,, \end{aligned} \tag{2.20}$$

where $h^{(\mathrm{NS})}(*,m)$, $Q^{(\mathrm{NS})}(m)$ are those given in (2.13), (2.16) and (2.18). Useful properties of the extended characters (2.12), (2.14) and (2.17) are summarized in appendix C.

The non-compactness of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ model leads to an IR divergence in the torus partition function. One may introduce an IR cut-off ϵ and then the regularized partition function contains a piece which consists of continuous representations and also a piece consisting of discrete representations [37]. Since the continuous representations describe string modes propagating in the bulk, their contributions are proportional to the volume of target space $V(\epsilon)$, while the discrete representations describe localized string states and their contributions are volume independent.

Leading terms in the infinite volume limit are given by continuous representations [4];

$$\begin{aligned} &\lim_{\epsilon\to+0}\frac{Z(\tau;\epsilon)}{V(\epsilon)}\propto \\ &\propto\frac{1}{2}\sum_{\sigma}\int_0^\infty dp\sum_{w\in 2K}\sum_{n\in N}\chi_{\mathbf{c}}{}^{(\sigma)}(p,Kn+Nw;\tau,0)\chi_{\mathbf{c}}{}^{(\sigma)}(p,-Kn+Nw;-\bar{\tau},0)\,. \end{aligned} \tag{2.21}$$

Here σ denotes the spin structure and the above partition function is modular invariant. The quantum numbers n, w are identified with the KK momenta and winding modes along the circle of the euclidean cigar geometry with an asymptotic radius $\sqrt{2k}\equiv\sqrt{2N/K}$ of the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$-coset theory [38].

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If one considers the $\widetilde{\mathrm{R}}$-part of the partition function, contributions of continuous representations drop out and only the discrete representations survive. They give rise to a volume-independent finite result. This is nothing but the Witten index;

$$
\begin{aligned}
Z^{(\widetilde{\mathrm{R}})}(\tau) &= \sum_{s=K}^{N+K} \sum_{w \in \mathbf{Z}_{2K}} \sum_{n \in \mathbf{Z}_N} a(s)\, {\chi_{\mathbf{d}}}^{(\widetilde{\mathrm{R}})}(s, Kn+Nw; \tau, 0) {\chi_{\mathbf{d}}}^{(\widetilde{\mathrm{R}})}(s, -Kn+Nw; -\bar{\tau}, 0)\,, \\
a(s) &\equiv \begin{cases} 1 & K+1 \leq s \leq N+K-1 \\ \frac{1}{2} & s = K, N+K \end{cases} \quad .
\end{aligned} \tag{2.22}
$$

It is important to note that the quantum number s runs over the range [4, 5] ;

$$
K \leq s \leq N+K, \tag{2.23}
$$

which is strictly smaller than (2.15) if $K \neq 1$ (see also [37]). This range is consistent with the modular transformation formulas (C.13), (C.14).

3. Non-compact cosets coupled to minimal models

Now, we work on the main subject in this paper. Let us study the superconformal system defined as

$$
\Big[L_{N_1,K_1} \otimes \cdots \otimes L_{N_{N_L},K_{N_L}} \otimes M_{k_1} \otimes \cdots \otimes M_{k_{N_M}} \Big]_{\mathrm{U}(1)\text{-projection}}\,, \tag{3.1}
$$

where $L_{N,K}$ denotes the $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$ Kazama-Suzuki model with $k = N/K$ ($\hat{c} = 1 + 2K/N$) and M_k denotes the level k $\mathcal{N}=2$ minimal model with $\hat{c} = k/(k+2)$. We impose the criticality condition for the case of target manifold with (complex) dimension $\mathbf{n}$

$$
\sum_{i=1}^{N_M} \frac{k_i}{k_i+2} + \sum_{j=1}^{N_L} \left(1 + \frac{2K_j}{N_j}\right) = \mathbf{n}\,, \qquad \mathbf{n} = 2, 3, 4\,. \tag{3.2}
$$

Since $\hat{c} > 1$ for each $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$-sector, it is obvious that $N_L \leq \mathbf{n}-1$ holds. We expect that the $\mathrm{U}(1)$-charge projection yields consistent superstring vacua describing non-compact $CY_{\mathbf{n}}$ compactifications with d-dimensional Minkowski space ($d = 10 - 2\mathbf{n}$). Note that the periodicity of extended characters depends on the choice of N_j, K_j, not only on the ratio N_j/K_j. We shall thus adopt the notation L_{N_j,K_j} to indicate which extended characters are used, although only the ratio N_j/K_j parameterizes the $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$ supercoset. For simplicity we here assume that each pair N_j, K_j is relatively prime for every $j = 1, \ldots, N_L$. We set

$$
N \equiv \text{L.C.M.}\,\{k_i+2,\, N_j\}\,, \qquad i = 1, \cdots, N_M, \qquad j = 1, \cdots, N_L\,, \tag{3.3}
$$

and then the required $\mathrm{U}(1)$-projection is reduced to the $\mathbb{Z}_N$-orbifoldization. We introduce the notations

$$
\mu_i, \nu_j \in \mathbb{Z}_{>0}\,, \qquad N = \mu_i(k_i+2) = \nu_j N_j\,, \tag{3.4}
$$

for later convenience.

3.1 Toroidal partition functions: continuous part of closed string spectra

We first analyse the closed string sector. Only the continuous part of closed string spectrum contributes to the modular invariant partition function (per unit volume), and should be interpreted as the propagating modes in the non-compact Calabi-Yau space. More detailed argument has been given in [4] for the case $N_L = 1$. Let us start by assuming

- diagonal modular invariance in each M_{k_i}-sector,
- the partition function (2.21) for each L_{N_j,K_j}-sector,

before performing the $\mathbb{Z}_N$-orbifoldization. As in the standard treatment of orbifold, the $\mathbb{Z}_N$-projection must be accompanied by the twisted sectors generated by the spectral flows. The integral spectral flows act on each character as the shifts of the angular variable; $z \mapsto z + a\tau + b$ ($a, b \in \mathbb{Z}_N$), and thus the relevant conformal blocks are defined as the flow invariant orbits [29];

$$
\begin{aligned}
\mathcal{F}^{(\mathrm{NS})}_{I,\mathbf{p},\mathbf{w}}(\tau, z) &\equiv \frac{1}{N} \sum_{a,b \in \mathbf{Z}_N} q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} a z} \prod_{i=1}^{N_M} \mathrm{ch}^{(\mathrm{NS})}_{\ell_i, m_i}(\tau, z + a\tau + b) \times \\
&\quad \times \prod_{j=1}^{N_L} \chi_{\mathbf{c}}{}^{(\mathrm{NS})}(p_j, K_j n_j + N_j w_j; \tau, z + a\tau + b)\,, \\
\tilde{\mathcal{F}}^{(\mathrm{NS})}_{I,\mathbf{p},\mathbf{w}}(-\bar{\tau}, \bar{z}) &\equiv \frac{1}{N} \sum_{a,b \in \mathbf{Z}_N} \bar{q}^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} a \bar{z}} \prod_{i=1}^{N_M} \mathrm{ch}^{(\mathrm{NS})}_{\ell_i, m_i}(-\bar{\tau}, \bar{z} + a\bar{\tau} + b) \times \\
&\quad \times \prod_{j=1}^{N_L} \chi_{\mathbf{c}}{}^{(\mathrm{NS})}(p_j, -K_j n_j + N_j w_j; -\bar{\tau}, -\bar{z} - a\bar{\tau} - b) \equiv \mathcal{F}^{(\mathrm{NS})}_{I,\mathbf{p},-\mathbf{w}}(-\bar{\tau}, \bar{z})\,, \\
I &= ((\ell_1, m_1), \ldots, (\ell_{N_M}, m_{N_M}), n_1, \ldots, n_{N_L})\,,\ \mathbf{p} = (p_1, \ldots, p_{N_L})\,,\ \mathbf{w} = (w_1, \ldots, w_{N_L})\,,
\end{aligned}
\tag{3.5}
$$

where $\mathrm{ch}^{(\mathrm{NS})}_{\ell_i,m_i}(\tau, z)$ denotes the character of the minimal model M_{k_i} and $\chi_{\mathbf{c}}{}^{(\mathrm{NS})}(p_j, K_j n_j + N_j w_j; \tau, z)$ is the extended character (2.12) of the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ theory L_{N_j,K_j}.[1] The summation $\displaystyle\frac{1}{N}\sum_{b \in \mathbf{Z}_N} *$ imposes the constraint on the U(1)-charge

$$
\sum_{i=1}^{N_M} \frac{m_i}{k_i + 2} + \sum_{j=1}^{N_L} \frac{K_j n_j}{N_j} \in \mathbb{Z}\,,
\tag{3.6}
$$

and we automatically have $\mathcal{F}^{(\mathrm{NS})}_{*} \equiv 0$ unless (3.6) is satisfied. The conformal blocks of other spin structures are defined by the 1/2-spectral flows;

$$
\begin{aligned}
\mathcal{F}^{(\widetilde{\mathrm{NS}})}_{I,\mathbf{p},\mathbf{w}}(\tau, z) &\equiv \mathcal{F}^{(\mathrm{NS})}_{I,\mathbf{p},\mathbf{w}}\left(\tau, z + \frac{1}{2}\right), \qquad \mathcal{F}^{(\mathrm{R})}_{I,\mathbf{p},\mathbf{w}}(\tau, z) \equiv q^{\frac{\mathbf{n}}{8}} e^{i\pi \mathbf{n} z} \mathcal{F}^{(\mathrm{NS})}_{I,\mathbf{p},\mathbf{w}}\left(\tau, z + \frac{\tau}{2}\right), \\
\mathcal{F}^{(\widetilde{\mathrm{R}})}_{I,\mathbf{p},\mathbf{w}}(\tau, z) &\equiv q^{\frac{\mathbf{n}}{8}} e^{i\pi \mathbf{n} z} \mathcal{F}^{(\mathrm{NS})}_{I,\mathbf{p},\mathbf{w}}\left(\tau, z + \frac{\tau}{2} + \frac{1}{2}\right).
\end{aligned}
\tag{3.7}
$$

[1] In the right-moving sector of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ theory we have chosen an angular dependence $-\bar{z}$. This is in order to bring our convention of quantum numbers n_j, w_j consistent with those given by the cigar geometry of 2-dimensional black hole.

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By construction the conformal blocks $\mathcal{F}^{(\sigma)}_{I,\mathbf{p},\mathbf{w}}$ have the following symmetry ($^\forall a, ^\forall b \in \mathbb{Z}$)

$$
\begin{aligned}
q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} a z} \mathcal{F}^{(\mathrm{NS})}_{I,\mathbf{p},\mathbf{w}}(\tau, z + a\tau + b) &= \mathcal{F}^{(\mathrm{NS})}_{I,\mathbf{p},\mathbf{w}}(\tau, z)\,, \\
q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} a z} \mathcal{F}^{(\widetilde{\mathrm{NS}})}_{I,\mathbf{p},\mathbf{w}}(\tau, z + a\tau + b) &= (-1)^{\mathbf{n}a} \mathcal{F}^{(\widetilde{\mathrm{NS}})}_{I,\mathbf{p},\mathbf{w}}(\tau, z)\,, \\
q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} a z} \mathcal{F}^{(\mathrm{R})}_{I,\mathbf{p},\mathbf{w}}(\tau, z + a\tau + b) &= (-1)^{\mathbf{n}b} \mathcal{F}^{(\mathrm{R})}_{I,\mathbf{p},\mathbf{w}}(\tau, z)\,, \\
q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i \mathbf{n} a z} \mathcal{F}^{(\widetilde{\mathrm{R}})}_{I,\mathbf{p},\mathbf{w}}(\tau, z + a\tau + b) &= (-1)^{\mathbf{n}(a+b)} \mathcal{F}^{(\widetilde{\mathrm{R}})}_{I,\mathbf{p},\mathbf{w}}(\tau, z)\,,
\end{aligned} \tag{3.8}
$$

Taking the diagonal modular invariance for the spin structures, we obtain the partition function (as a non-linear σ-model)

$$
Z(\tau, z) = e^{-2\pi \mathbf{n} \frac{(\mathrm{Im}\, z)^2}{\tau_2}} \frac{1}{2N} \sum_{\sigma} \sum_{I,\mathbf{w}} \int d^{N_L}\mathbf{p}\, \mathcal{F}^{(\sigma)}_{I,\mathbf{p},\mathbf{w}}(\tau, z) \mathcal{F}^{(\sigma)}_{I,\mathbf{p},-\mathbf{w}}(-\bar{\tau}, \bar{z})\,. \tag{3.9}
$$

The overall factor $1/N$ is necessary to avoid the overcounting of states. One can easily check the modular invariance of this partition function using the modular properties of L_{N_j,K_j}, M_{k_i} given in appendices. A crucial point is the fact that the sum over the spectral flow $z \mapsto z + a\tau + b$ in (3.5) preserves modular invariance.

Incorporating the flat space-time $\mathbb{R}^{d-1,1}$ (with $d/2 + \mathbf{n} = 5$), we also obtain the supersymmetric conformal blocks of superstring vacua

$$
\frac{1}{\tau_2^{\frac{d-2}{4}} \eta(\tau)^{d-2}} \sum_{\sigma} \epsilon(\sigma) \left(\frac{\theta_{[\sigma]}(\tau)}{\eta(\tau)}\right)^{\frac{d-2}{2}} \mathcal{F}^{(\sigma)}_{I,\mathbf{p},\mathbf{w}}(\tau, 0)\,, \tag{3.10}
$$

where $\theta_{[\sigma]}$ denotes θ_3, θ_4, θ_2, $i\theta_1$ for $\sigma = \mathrm{NS}, \widetilde{\mathrm{NS}}, \mathrm{R}, \widetilde{\mathrm{R}}$ respectively, and we set $\epsilon(\mathrm{NS}) = \epsilon(\widetilde{\mathrm{R}}) = +1$, $\epsilon(\widetilde{\mathrm{NS}}) = \epsilon(\mathrm{R}) = -1$. The conformal blocks (3.10) actually vanish for arbitrary τ [39].

One can choose a large variety of modular invariants as consistent conformal theories. For example, one may take general modular invariants of the types given in [40] with respect to $\mathbf{w} \in \mathbb{Z}_{2K_1} \times \cdots \times \mathbb{Z}_{2K_{N_L}}$.

It turns out that some of the familiar non-compact Calabi-Yau spaces do not correspond to the simplest choice of modular invariance (3.9) and we have to use a somewhat more non-trivial form of modular invariant. A typical example showing such peculiar feature is the singular CY_3 of A_{n-1}-type ($CY_3(A_{n-1})$) [30, 20]. The conformal blocks for this model presented in [22] are written (with suitable change of notations) as

$$
\begin{aligned}
\mathcal{F}^{(\mathrm{NS})}_{\ell,w}(\tau, z) &= \sum_{m \in \mathbf{Z}_{4n}} \mathrm{ch}^{(\mathrm{NS})}_{\ell,m}(\tau, z) \frac{\Theta_{-(n+2)m+2nw,2n(n+2)}\left(\tau, \frac{z}{n}\right)}{\eta(\tau)}\,, \\
&\ell + 2w \in 2\mathbb{Z}\,, \qquad (w \in \tfrac{1}{2}\mathbb{Z}_{4(n+2)})\,,
\end{aligned} \tag{3.11}
$$

where we omitted the factor depending on the 'Liouville momentum' p. These are identified with the branching functions of the coset CFT: $\dfrac{\mathrm{SU}(2)_{n-2} \times \mathrm{SO}(4)_1}{\mathrm{U}(1)_{n+2}}$. At first glance, (3.11) seems to fit with the formulas (3.5) with $N = 2n$, $K = n+2$. However, *half-integral* values of w are now allowed with the constraint

$$m + 2w \in 2\mathbb{Z}\,. \tag{3.12}$$

This condition may be interpreted as some kind of orbifoldization, and makes it possible to pair each of the primary states of the minimal model M_{n-2} to those of $L_{2n,n+2}$ so that it yields a physical state with an integral U(1)-charge. As we shall see later, we obtain the expected spectrum of massless states as the singular CY_3 of A_{n-1}-type and the correct open string Witten indices under this condition (3.12). We denote the $\mathrm{SL}(2;\mathbb{R})/\,\mathrm{U}(1)$-sector defined this way as $L'_{2n,n+2}$ from now on.

A similar example which we will later study is a model with two Liouville fields $N_L = 2$ and $N_M = 1$;

$$\hat{c} = 4\,, \qquad k_1 = n-2\,, \qquad N_1 = N_2 = 4n\,, \qquad K_1 = K_2 = n+2\,. \tag{3.13}$$

It is possible to show that the following conformal blocks give the consistent superstring vacuum;

$$\begin{aligned}
\mathcal{F}^{(\mathrm{NS})}_{\ell,m,m_j,w_j,p_j}(\tau,z) &= \frac{1}{4n}\sum_{a\in\mathbf{Z}_{4n}} \mathrm{ch}^{(\mathrm{NS})}_{\ell,m-2a}(\tau,z)\times \\
&\quad\times \prod_{j=1,2}\Big\{\chi_{\mathbf{c}}{}^{(\mathrm{NS})}_{(4n,n+2)}(p_j,(n+2)m_j+4nw_j+2(n+2)a;\tau,z) + \\
&\qquad\qquad + \chi_{\mathbf{c}}{}^{(\mathrm{NS})}_{(4n,n+2)}(p_j,(n{+}2)(m_j{+}4n)+4nw_j{+}2(n{+}2)a;\tau,z)\Big\} \\
&\equiv \frac{1}{2n}\sum_{a\in\mathbf{Z}_{2n}} \mathrm{ch}^{(\mathrm{NS})}_{\ell,m-2a}(\tau,z)\times \\
&\quad\times \prod_{j=1,2}\chi_{\mathbf{c}}{}^{(\mathrm{NS})}_{(2n,\frac{n+2}{2})}\left(p_j,\frac{n+2}{2}m_j+2nw_j+(n+2)a;\tau,z\right) \\
&\quad(\text{if } n \text{ is even})\,,
\end{aligned} \tag{3.14}$$

with

$$m\in\mathbb{Z}_{2n}\,,\quad m_j\in\mathbb{Z}_{4n}\,,\quad w_j\in\frac{1}{4}\mathbb{Z}_{4(n+2)}\,,\quad m_j+4w_j\in 2\mathbb{Z}\,,\quad \sum_{j=1,2}(m_j+4w_j)\in 4\mathbb{Z}\,, \tag{3.15}$$

and the U(1)-charge condition

$$2m + m_1 + m_2 \in 2n\mathbb{Z}\,. \tag{3.16}$$

We here use the notation $\chi_{\mathbf{c}}{}^{(*)}_{(N,K)}(*,*;\tau,z)$ with the parameters N, K written explicitly. By careful calculations it is possible to show that the conformal blocks (3.14) are in fact closed under modular transformations in a manner consistent with the non-trivial restrictions (3.15), (3.16). The coefficients of S-transformation include the factors

$$\frac{1}{\sqrt{4n}}e^{-2\pi i\frac{m_j m'_j}{4n}}\cdot\frac{1}{\sqrt{4(n+2)}}e^{2\pi i\frac{4w_j w'_j}{n+2}}$$

in each $L'_{4n,n+2}$-sector, and we can construct modular invariants in the standard way. Hence this model yields a consistent string vacuum with the choice of spectrum (3.15). We denote the system defined this way as $L'_{4n,n+2}$. All the primaries in M_{n-2} can again find partners in two $L'_{4n,n+2}$-sectors. We will later identify this vacuum as a non-compact CY_4 with a singular CY_3 fibered over $\mathbb{C}P^1$.

3.2 Elliptic genera: discrete part of closed string spectra

Let us turn to the discrete spectrum in the closed string Hilbert space. It describes localized string excitations and is of basic importance since massless states appear in this sector. A useful quantity that captures the BPS states is the elliptic genus, and we try to evaluate it for general models (3.1). It is basically a generalization of the analysis given in [4] to the case $N_L \geq 1$. We consider flow invariant orbits consisting of the discrete characters (2.14) in place of (3.5);

$$\begin{aligned}
\mathcal{G}^{(\mathrm{NS})}_{I,\mathbf{s},\mathbf{w}}(\tau,z) &\equiv \frac{1}{N}\sum_{a,b\in\mathbf{Z}_N} q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i\mathbf{n}az} \prod_{i=1}^{N_M} \mathrm{ch}^{(\mathrm{NS})}_{\ell_i,m_i}(\tau, z+a\tau+b)\ \times \\
&\quad \times \prod_{j=1}^{N_L} {\chi_{\mathbf{d}}}^{(\mathrm{NS})}(s_j, K_j n_j + N_j w_j; \tau, z+a\tau+b)\,, \\
\tilde{\mathcal{G}}^{(\mathrm{NS})}_{I,\mathbf{s},\mathbf{w}}(-\bar\tau,\bar z) &\equiv \frac{1}{N}\sum_{a,b\in\mathbf{Z}_N} \bar q^{\frac{\mathbf{n}}{2}a^2} e^{2\pi i\mathbf{n}a\bar z} \prod_{i=1}^{N_M} \mathrm{ch}^{(\mathrm{NS})}_{\ell_i,m_i}(-\bar\tau, \bar z+a\bar\tau+b) \times \\
&\quad \times \prod_{j=1}^{N_L} {\chi_{\mathbf{d}}}^{(\mathrm{NS})}(s_j, -K_j n_j + N_j w_j; -\bar\tau, -\bar z-a\bar\tau-b) \\
&\equiv \mathcal{G}^{(\mathrm{NS})}_{I,\widehat{\mathbf{s}},\widehat{\mathbf{w}}}(-\bar\tau,\bar z)\,, \\
I &= ((\ell_1,m_1),\ldots,(\ell_{N_M},m_{N_M}),n_1,\ldots,n_{N_L})\ , \\
\mathbf{s} &= (s_1,\ldots,s_{N_L})\,, \qquad \mathbf{w} = (w_1,\ldots,w_{N_L})\,, \\
\widehat{\mathbf{s}} &= (N_1+2K_1-s_1,\ldots,N_{N_L}+2K_{N_L}-s_{N_L})\,, \\
\widehat{\mathbf{w}} &= (1-w_1,\ldots,1-w_{N_L})\,,
\end{aligned} \tag{3.17}$$

Recall the charge conjugation relations (C.7) to derive the last equality. In the limit $z\to 0$ we obtain the Witten index

$$\lim_{z\to 0}\ \mathcal{G}^{(\widetilde{\mathrm{R}})}_{I,\mathbf{s},\mathbf{w}}(\tau,z) \equiv \mathcal{I}_{I,\mathbf{s},\mathbf{w}}\,, \tag{3.18}$$

which can be evaluated by using the formulas (B.16) and (C.4). The elliptic genus is then written as

$$\mathcal{Z}(\tau,z) = \frac{1}{N}\sum_{I,\mathbf{s},\mathbf{w}} \mathbf{a}(\mathbf{s})\mathcal{I}_{I,\widehat{\mathbf{s}},\widehat{\mathbf{w}}}\mathcal{G}^{(\widetilde{\mathrm{R}})}_{I,\mathbf{s},\mathbf{w}}(\tau,z)\,, \tag{3.19}$$

where we set

$$\mathbf{a}(\mathbf{s}) = \prod_{j=1}^{N_L} a(s_j)\,, \qquad a(s_j) = \begin{cases} 1 & K_j+1\leq s_j\leq N_j+K_j-1 \\ \dfrac{1}{2} & s_j = K_j,\ N_j+K_j. \end{cases} \tag{3.20}$$

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In the cases of CY_3 ($\mathbf{n}=3$) the elliptic genera are shown to have a particularly simple form;

$$\mathcal{Z}(\tau,z) = \frac{\chi}{2}\frac{\theta_1(\tau,2z)}{\theta_1(\tau,z)}\,, \tag{3.21}$$

$$\chi = \frac{1}{N}\sum_{I,\mathbf{s},\mathbf{w}} \mathbf{a}(\mathbf{s})\mathcal{I}_{I,\mathbf{s},\mathbf{w}}\mathcal{I}_{I,\widehat{\mathbf{s}},\widehat{\mathbf{w}}}\,. \tag{3.22}$$

Let us next try to exhibit more explicit forms of elliptic genera. To this end it is useful to recall the formula of elliptic genus of minimal model [41]

$$\mathcal{Z}_k(\tau,z) = \sum_{\ell=0}^{k} \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell,\ell+1}(\tau,z) = -\sum_{\ell=0}^{k} \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell,-(\ell+1)}(\tau,z) = \frac{\theta_1(\tau,\frac{k+1}{k+2}z)}{\theta_1(\tau,\frac{1}{k+2}z)}\,. \tag{3.23}$$

The corresponding formula for the $L_{N,K}$-sector is written as [4][2]

$$\begin{aligned}\mathcal{Z}_{N,K}(\tau,z) &\equiv \sum_{s=K}^{N+K} a(s)\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s,s-K;\tau,z)\\ &\equiv \left[\mathcal{K}_{2NK}\left(\tau,\frac{z}{N},0\right) - \frac{1}{2}\Theta_{0,NK}\left(\tau,\frac{2z}{N}\right)\right]\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,,\end{aligned} \tag{3.24}$$

where $\mathcal{K}_\ell(\tau,\nu,\mu)$ is the level ℓ Appell function [33, 34] defined by

$$\mathcal{K}_\ell(\tau,\nu,\mu) \equiv \sum_{m\in\mathbf{Z}} \frac{e^{i\pi m^2\ell\tau+2\pi i m\ell\nu}}{1-e^{2\pi i(\nu+\mu+m\tau)}}\,. \tag{3.25}$$

The following identity is quite useful;

$$\begin{aligned}&\sum_{s=K}^{N+K-1} e^{2\pi i\frac{(s-K)b}{N}}\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s,s-K+2Ka;\tau,z) =\\ &\quad = q^{\frac{K}{N}a^2}y^{\frac{2K}{N}a}\mathcal{K}_{2NK}\left(\tau,\frac{z+a\tau+b}{N},0\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,,\quad (a,b\in\mathbb{Z}_N)\,,\end{aligned} \tag{3.26}$$

or conversely,

$$\begin{aligned}&\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s,s-K+2Ka;\tau,z) =\\ &\quad = \frac{1}{N}\sum_{b\in\mathbf{Z}_N} e^{-2\pi i\frac{(s-K)b}{N}} q^{\frac{K}{N}a^2}y^{\frac{2K}{N}a}\mathcal{K}_{2NK}\left(\tau,\frac{z+a\tau+b}{N},0\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,,\\ &\quad (a\in\mathbb{Z}_N,\quad K\le s\le N+K-1)\,.\end{aligned} \tag{3.27}$$

One may regard these relations as non-compact analogue of the formula (3.23). More details on the relation between extended characters and Appell functions are discussed in appendix E.

[2]Overall sign is opposite to that of [4].

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Our goal is to derive the "orbifold forms" of elliptic genera like those given in [42]. To this end we have to slightly modify (3.24) except for the cases of $N_M = N_L = 1$ treated in [4], so as to correctly reproduce (3.19). We define

$$\begin{aligned} \widehat{\mathcal{Z}}_{N,K}(\tau,z) &\equiv \sum_{s=K}^{N+K} a(s)\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s,s-K-2Kn(s);\tau,z) \\ &\equiv \left[\frac{1}{N}\sum_{s=K}^{N+K-1}\sum_{b\in\mathbf{Z}_N} e^{-2\pi i\frac{(s-K)b}{N}} q^{\frac{K}{N}n(s)^2} y^{-\frac{2K}{N}n(s)} \mathcal{K}_{2NK}\left(\tau,\frac{z-n(s)\tau+b}{N},0\right) - \right. \\ &\qquad \left. -\frac{1}{2}\Theta_{0,NK}\left(\tau,\frac{2z}{N}\right)\right]\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}, \end{aligned} \tag{3.28}$$

where $n(s)$ is defined uniquely by the condition

$$Kn(s) \equiv s-K \ (\mathrm{mod}\, N), \qquad n(s)\in\mathbb{Z}_N. \tag{3.29}$$

(This is well-defined for each s, since we are assuming that N and K are relatively prime.) In the special case $K=1$, we simply have $n(s)=s-1$. The elliptic genus (3.19) is now rewritten as the orbifold form;

$$\begin{aligned} \mathcal{Z}(\tau,z) &= \frac{1}{N}\sum_{a,b\in\mathbf{Z}_N} (-1)^{(N_M+N_L)(a+b)} q^{\frac{\mathbf{n}}{2}a^2} y^{\mathbf{n}a} \times \\ &\qquad \times \prod_{i=1}^{N_M}\mathcal{Z}_{k_i}(\tau,z+a\tau+b)\prod_{j=1}^{N_L}\widehat{\mathcal{Z}}_{N_j,K_j}(\tau,z+a\tau+b). \end{aligned} \tag{3.30}$$

For example, in the case of compactification on $ALE(A_{n-1})$ spaces (i.e. $N_M = N_L = 1$, $k=n-2$, $N=n$, $K=1$), the formula (3.30) is reduced to

$$\begin{aligned} \mathcal{Z}_{ALE(A_{n-1})}(\tau,z) &= \sum_{\ell=0}^{n-2}\sum_{r\in\mathbf{Z}_n} \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell,\ell+1-2r}(\tau,z)\,\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(\ell+2,\ell+3-2(\ell+2)+2r;\tau,z) \\ &= \sum_{\ell=0}^{n-2}\sum_{r\in\mathbf{Z}_n} \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell,-(\ell+1)-2r}(\tau,z)\,\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(\ell+2,\ell+1+2r;\tau,z), \end{aligned} \tag{3.31}$$

which reproduces the one given in [4].

3.3 Massless closed string spectra

It is an important task to analyze the massless closed string spectrum. We can solve this problem in a similar manner as in the compact Gepner models, since we have already constructed the conformal blocks in closed string sector. Massless states correspond to the (anti-) chiral primary states of conformal weights $h=\tilde{h}=1/2$. As we discuss below, basic aspects of massless spectra in the non-compact models are summarized as follows:

- In the cases of $\hat{c}\neq 2$, there exists at most one chiral primary of the (a,c) (or (c,a))-type with $h=\tilde{h}=1/2$ in each spectral flow orbit (3.17), and none of the (c,c) (or (a,a))-type exist.[3]

[3] Of course, in mirror models where L_{N_j,K_j}-sectors are realized by $\mathcal{N}=2$ Liouville theories, the situation is reversed: no chiral primaries of (c,a) and (a,c)-types exist, while (c,c) and (a,a)-types are possible.

- In the cases of $\hat{c}=2$, we have at most a quartet of the (c,c), (a,a), (c,a) and (a,c)-type primaries with $h=\tilde{h}=1/2$ in each spectral flow orbit.

This fact implies that at our non-compact Gepner points for CY_3 or CY_4 moduli space there exist only the deformations of Kähler structure but not the deformations of complex structure. On the other hand, in the case of $K3$ surfaces $\hat{c}=2$ the superconformal symmetry is extended to $\mathcal{N}=4$ and the above massless states compose the spin (1/2,1/2) representation of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ of the $\mathcal{N}=4$ SCA. From the space-time point of view the quartet corresponds to a scalar (tensor)-multiplet of $(1,1)$ $((2,0))$ SUSY in 6 dimensions.

Let us consider the (a,c)-type massless states: the analysis for the (c,a)-type is parallel. We start with working on the left-moving sectors. The anti-chiral states are described by the conditions;

M_{k_i}-sector:

$$m_i=-\ell_i\,,\qquad (0\le \ell_i\le k_i)\,,\qquad \text{U(1)-charge}\ \ Q_i=-\frac{\ell_i}{k_i+2}\,, \tag{3.32}$$

L_{N_j,K_j}-sector:

$$\begin{aligned}&s_j=K_jn_j+N_jw_j+2K_j\,,\qquad (K_j\le s_j\le N_j+K_j)\,,\qquad m_j=K_jn_j+N_jw_j\,,\\ &\text{U(1)-charge}\ \ Q_j=\frac{K_jn_j+N_jw_j-N_j}{N_j}\,,\end{aligned} \tag{3.33}$$

and we must impose

$$-\sum_i\frac{\ell_i}{k_i+2}+\sum_j\frac{K_jn_j+N_jw_j-N_j}{N_j}=-1\,. \tag{3.34}$$

We now recall the left-right pairing of states in our construction is given as (see (3.17));

$$\begin{array}{ccc}\text{left-moving} & & \text{right-moving}\\ (\ell_i,m_i) & \Longleftrightarrow & (\tilde{\ell}_i=\ell_i,\tilde{m}_i=m_i)\\ (s_j,m_j=K_jn_j+N_jw_j) & \Longleftrightarrow & (\tilde{s}_j=N_j+2K_j-s_j,\tilde{m}_j=K_jn_j-N_jw_j+N_j)\end{array} \tag{3.35}$$

Thus before applying the spectral flow the right-moving state corresponding to (3.32), (3.33) has the following quantum numbers

$$(\tilde{\ell}_i,\tilde{m}_i)=(\ell_i,-\ell_i)\,,\qquad (\tilde{s}_j,\tilde{m}_j)=(-K_jn_j-N_jw_j+N_j,K_jn_j-N_jw_j+N_j)\,. \tag{3.36}$$

They have the total U(1)-charge

$$-\sum_i\frac{\ell_i}{k_i+2}+\sum_j\frac{K_jn_j-N_jw_j+N_j}{N_j}\,, \tag{3.37}$$

which is integer because of the constraint (3.34).

Now, we look for chiral or anti-chiral primaries with $\tilde{h}=1/2$ in the orbit of spectral flow starting from the states (3.36). The flow $\bar{z}\to\bar{z}+\bar{\tau}$ acts on the quantum numbers as

$$\tilde{m}_i\ \longrightarrow\ \tilde{m}_i-2\,,\qquad \tilde{m}_j\ \longrightarrow\ \tilde{m}_j+2K_j\,, \tag{3.38}$$

and it has periodicities k_i+2 and N_j respectively in the M_{k_i} and L_{N_j,K_j}-sectors. We find that only the orbits satisfying the condition

$$
{}^{\exists}r \in \mathbb{Z}_N\,, \text{ s.t. } \begin{cases} \ell_i \equiv r \pmod{k_i+2}\,, & {}^{\forall}i \\ n_j \equiv r \pmod{N_j}\,, & {}^{\forall}j \end{cases} \tag{3.39}
$$

contains a (unique) chiral state with the total U(1)-charge $Q_{\text{tot}}=1$;

$$
(\tilde{\ell}_i,\tilde{m}_i)=(\ell_i,\ell_i)\,,\quad (\tilde{s}_j,\tilde{m}_j)=(-K_jn_j+N_j(1-w_j),-K_jn_j+N_j(1-w_j))\,,\ ({}^{\forall}i,{}^{\forall}j)\,. \tag{3.40}
$$

It also contains a (unique) anti-chiral state with $Q_{\text{tot}}=1-\hat{c}$;

$$
\begin{aligned}
(\tilde{\ell}_i,\tilde{m}_i) &= (\ell_i,\ell_i+2)\cong(k_i-\ell_i,\ell_i-k_i)\,,\\
(\tilde{s}_j,\tilde{m}_j) &= (-K_jn_j+N_j(1-w_j),-K_j(n_j+2)+N_j(1-w_j))\,,\ ({}^{\forall}i,{}^{\forall}j)\,.
\end{aligned} \tag{3.41}
$$

In fact, (3.40), (3.41) are generated by the spectral flows $\bar{z}\to\bar{z}-r\bar{\tau}$, $\bar{z}\to\bar{z}-(r+1)\bar{\tau}$ from (3.36) respectively if (3.39) holds. Note that in the cases of $\hat{c}=3,4$ only (3.40) yields a massless state, while both of (3.40), (3.41) become massless states at $\hat{c}=2$.

Therefore, the spectrum of massless closed string states is given by the solutions (ℓ_i,n_j,w_j) $(0\le\ell_i\le k_i,\ n_j\in\mathbb{Z}_{N_j},\ w_j\in\mathbb{Z}_{2K_j})$ of the constraints

$$
\left(\sum_i\frac{\ell_i}{k_i+2}-\sum_j\frac{K_jn_j}{N_j}\right)=1+\sum_j(w_j-1)\,, \tag{3.42}
$$

$$
-K_j\le K_jn_j+N_jw_j\le N_j-K_j\,, \tag{3.43}
$$

as well as (3.39). The second condition (3.43) follows from the constraint $K_j\le s_j\le N_j+K_j$. Obviously, the counting of (c,a)-type chiral states can be carried out in the same way, yielding the equal number of massless states. We have thus shown the characteristic feature of massless states announced before.

We now present concrete examples that have clear geometrical interpretations.

1. Cases of one Liouville field $N_L=1$:

We first present examples with $N_M=N_L=1$. The condition (3.39) simply yields $\ell_1=n_1(\equiv\ell)$ in these cases.

1.1. ALE(A_{n-1}): $M_{n-2}\otimes L_{n,1}$ In this case the constraint (3.42) simply gives $w_1=0$, and (3.43) is equivalent with

$$
-1\le\ell\le n-1\,. \tag{3.44}
$$

We thus conclude that each (anti-)chiral primary state in the range $0\le\ell\le n-2$ in M_{n-2} can be paired up to massless states of the types of (c,c), (a,a), (c,a) and (a,c)-types of $M_{n-2}\otimes L_{n,1}$ theory.

1.2. CY_4 **(A_{n-1}):** $M_{n-2}\otimes L_{n,n+1}$ The condition (3.42) is solved as

$$
\ell=-w_1\,,\qquad w_1\in\mathbb{Z}_{2(n+1)}\,. \tag{3.45}
$$

(3.43) then gives

$$-(n+1) \leq \ell \leq -1\,, \tag{3.46}$$

which has no solution in the range $0 \leq \ell \leq n-2$. Therefore, we have no massless states in this case.

1.3. CY_3 **(**A_{n-1}**):** $M_{n-2} \otimes L'_{2n,n+2}$ This case is non-trivial. We have $N_1 = 2n$, $K_1 = n+2$, which are not necessarily relatively prime. As addressed before, we must allow the half-integral winding numbers w_1, and impose the constraint $n_1 + 2w_1 \in 2\mathbb{Z}$.

The constraint (3.42) now leads to

$$\ell = -2w_1\,, \qquad \begin{cases} w_1 \in \mathbb{Z}_{2(n+1)} & \ell \ : \ \text{even} \\ w_1 \in \frac{1}{2} + \mathbb{Z}_{2(n+1)} & \ell \ : \ \text{odd} \end{cases} \tag{3.47}$$

and (3.43) gives

$$-(n+2) \leq 2\ell \leq n-2\,. \tag{3.48}$$

We thus find that the (anti-)chiral states in M_{n-2} with

$$\ell = 0, 1, \ldots, \left[\frac{n-2}{2}\right] \tag{3.49}$$

produce massless states of the (c,a) and (a,c)-types.

As already mentioned in [4], these aspects of massless states in the above three examples are consistent with the spectra of *normalizable* chiral operators describing the moduli of vacua discussed in [30, 20, 21]. Especially, the third example correctly reproduces the spectrum of scaling operators in the $\mathcal{N}=2$ $SCFT_4$ of Argyres-Douglas points [43]. We also point out that the massless spectra here are consistent with the ones deduced from the "LSZ poles" in correlation functions presented in [44].

2. Cases of two Liouville fields $N_L = 2$**:**

2.1. $\hat{c} = 3$**,** $M_{n-2} \otimes L_{2n,1} \otimes L_{2n,1}$**:** The criticality condition is satisfied as

$$\hat{c} = \frac{n-2}{n} + \left(1 + \frac{1}{n}\right) + \left(1 + \frac{1}{n}\right) = 3\,. \tag{3.50}$$

This type of superconformal system has been first studied in [23] and proposed to be the superstring vacuum corresponding to the Seiberg-Witten theory with SU(n) gauge group without matter in the low energy regime. Geometrically the theory is supposed to describe space-time of the $ALE(A_{n-1})$-fibration over $\mathbb{C}P^1$, or the n NS5-branes wrapped around $\mathbb{C}P^1$ in the T-dual picture.

We have two possibilities of satisfying (3.39);

(i) $\ell_1 = n_1 = n_2 \equiv r$, $0 \leq r \leq n-2$,

(ii) $\ell_1 + n = n_1 = n_2 \equiv r$, $n \leq r \leq 2n-2$.

In the case (i), (3.42) gives $w_1 + w_2 = 1$, and (3.43) leads to

$$-1 \le r + 2nw_i \le 2n - 1\,, \qquad (i = 1, 2)\,. \tag{3.51}$$

There are no solutions to these constraints.

In the case (ii), (3.42) gives $w_1 + w_2 = 0$. If (and only if) we set $w_1 = w_2 = 0$, (3.51) are satisfied for arbitrary $n \le r \le 2n - 2$. We thus find that

$$\ell_1 = 0, 1, \dots, n-2\,, \qquad n_1 = n_2 = \ell_1 + n\,, \qquad w_1 = w_2 = 0\,, \quad \Longleftrightarrow \text{ massless states}\,. \tag{3.52}$$

The (c, a)-type chiral fields also gives the equal number of massless states. These are identified as the moduli $u_2, \dots, u_n$ in the SU(n) SW theory. Especially, the marginal deformation for $\ell = 0$ corresponds to the size of base $\mathbb{C}P^1$ as suggested in [26].

2.2. $\hat{c} = 3$**,** $M_{n-2} \otimes L_{n\mu,K_1} \otimes L_{n\mu,K_2}$**,** $K_1 + K_2 = \mu$**, G.C.D**$\{K_i\} = 1$**:** This is a natural generalization of the example **2.1**. The criticality condition is satisfied as

$$\hat{c} \equiv \left(1 - \frac{2}{n}\right) + \left(1 + \frac{2K_1}{N_1}\right) + \left(1 + \frac{2K_2}{N_2}\right) = 3 + \frac{2}{N}\left(-\mu + K_1 + K_2\right) = 3\,. \tag{3.53}$$

In this case, using the relation $K_1 + K_2 = \mu$, we again obtain

$$\ell_1 = 0, 1, \dots, n-2\,, \qquad n_1 = n_2 = \ell_1 + n\,, \qquad w_1 = w_2 = 0\,, \quad \Longleftrightarrow \text{ massless states}\,. \tag{3.54}$$

This type of string vacua are identified as the non-compact CY_3 with the structure of $ALE(A_{n-1})$-fibration over the weighted projective space $W\mathbb{C}P^1\,[K_1, K_2]$.

2.3. $\hat{c} = 4$**,** $M_{n-2} \otimes L'_{4n,n+2} \otimes L'_{4n,n+2}$**:** We next consider a more subtle example. The criticality condition is satisfied as

$$\hat{c} = \frac{n-2}{n} + \left(1 + \frac{n+2}{2n}\right) + \left(1 + \frac{n+2}{2n}\right) = 4\,. \tag{3.55}$$

Similarly to the case of $CY_3(A_{n-1})$, we have to allow $w_j \in \frac{1}{4}\mathbb{Z}$ and assume (3.15) in each of $L'_{4n,n+2}$-sector in order to obtain the expected spectrum. As in the example 2.1, massless states are possible only for

$$\ell_1 + n = n_1 = n_2 = r\,, \qquad 0 \le r \le 2n - 2\,, \qquad r + 4w_i = 0\,, \tag{3.56}$$

and (3.43) gives us

$$\ell = 0, 1, \dots, \left[\frac{n-2}{2}\right] \quad \Longleftrightarrow \text{ massless states}\,. \tag{3.57}$$

This spectrum is the same as $CY_3\ (A_{n-1})$. We propose that this model is identified as $CY_3(A_{n-1})$-fibration on $\mathbb{C}P^1$.

The generalizations similar to the example **2.2** are straightforward; $M_{n-2} \otimes L'_{2n\mu,K_1(n+2)} \otimes L'_{2n\mu,K_2(n+2)}$, $K_1 + K_2 = \mu$, G.C.D$\{K_i\} = 1$. It is expected that it describes the $CY_3(A_{n-1})$-fibration over $W\mathbb{C}P^1\,[K_1, K_2]$ and we again obtain the same massless spectrum (3.49).

3. Cases of three Liouville fields $N_L = 3$:

The $\hat{c} = 4$ vacua are the only possibility for these cases.

3.1. $M_{n-2} \otimes L_{3n,1} \otimes L_{3n,1} \otimes L_{3n,1}$**:** The criticality condition is satisfied as

$$\hat{c} = \frac{n-2}{n} + 3 \times \left(1 + \frac{2}{3n}\right) = 4\,. \tag{3.58}$$

The massless states are possible only for

$$\ell_1 + 2n = n_1 = n_2 = n_3 = r\,, \qquad 2n \leq r \leq 3n-2\,, \qquad w_1 = w_2 = w_3 = 0\,. \tag{3.59}$$

(3.43) gives us

$$\ell_1 = 0, 1, \ldots, n-2 \iff \text{massless states}\,, \tag{3.60}$$

which is the same spectrum as that of $ALE(A_{n-1})$. However, we can only have the (a, c) and (c, a)-type massless chiral states contrary to the ALE case. This model is identified as the $ALE(A_{n-1})$-fibration over $\mathbb{C}P^2$ [26].

The generalization similar to the example **2.2** is also easy; $M_{n-2} \otimes L_{n\mu,K_1} \otimes L_{n\mu,K_2} \otimes L_{n\mu,K_3}$, $K_1 + K_2 + K_3 = \mu$, G.C.D$\{K_i\} = 1$. This model is identified as the $ALE(A_{n-1})$-fibration over $W\mathbb{C}P^2\,[K_1, K_2, K_3]$ and we again obtain the same massless spectrum (3.60).

3.4 Notes on geometrical interpretations

Let us here clarify our geometrical interpretation of the string vacua considered above as non-compact Calabi-Yau spaces. We first recall the familiar CY/LG correspondence:

$$X_1^{r_1} + \cdots + X_{n+2}^{r_{n+2}} = 0\,, \qquad \text{in } W\mathbb{C}P_{n+1}\left[\frac{1}{r_1}, \ldots, \frac{1}{r_{n+2}}\right]\,, \tag{3.61}$$

defines a Calabi-Yau n-fold if $\sum_{i=1}^{n+2} \frac{1}{r_i} = 1$, and is equivalent to the LG orbifold defined by the superpotential $W(\mathbf{X}_i) \equiv \mathbf{X}_1^{r_1} + \cdots + \mathbf{X}_{n+2}^{r_{n+2}}$, where $\mathbf{X}_i$ denotes the chiral superfields. If some of r_i are negative, the corresponding Calabi-Yau space becomes non-compact. In fact, such LG interpretation was the starting point for the CFT descriptions of non-compact Calabi-Yau spaces [17, 18, 20, 23], and was further refined from the viewpoints of mirror symmetry in [26].

As an illustration, we consider the example **2.1**: $\hat{c} = 3$, $M_{n-2} \otimes L_{2n,1} \otimes L_{2n,1}$. The corresponding LG model is given as [23]

$$W = \mathbf{X}^n + \mathbf{Y}_1^{-2n} + \mathbf{Y}_2^{-2n}\,, \tag{3.62}$$

which describes the non-compact CY space

$$X^n + Y_1^{-2n} + Y_2^{-2n} + w_1^2 + w_2^2 = 0\,, \qquad \text{in} \quad W\mathbb{C}P^4\,[2, -1, -1, n, n]\,. \tag{3.63}$$

This formula has structure of the $ALE(A_{n-1})$-fibration over $\mathbb{C}P^1$. Following [20], one may rewrite (3.62) as the Liouville form;[4]

$$W = \mathbf{X}^n + e^{-\frac{1}{\mathcal{Q}}\mathbf{X}_1} + e^{-\frac{1}{\mathcal{Q}}\mathbf{X}_2}\,, \tag{3.64}$$

where we set $\mathcal{Q} = \sqrt{1/n}$. This realization amounts to expressing the $L_{2n,1}$-sectors as the $\mathcal{N} = 2$ Liouville theories, and the linear dilaton is given by

$$\Phi = -\frac{\mathcal{Q}}{2}\mathrm{Re}\,(\mathbf{X}_1 + \mathbf{X}_2)\,. \tag{3.65}$$

Equivalently, one may rewrite it as

$$W = \mathbf{X}^n + e^{-n\mathbf{Z}}(e^{\mathbf{Y}} + e^{-\mathbf{Y}}), \tag{3.66}$$

where we set $\mathbf{X}_1 = n\mathcal{Q}\mathbf{Z} + \mathcal{Q}\mathbf{Y}$, $\mathbf{X}_2 = n\mathcal{Q}\mathbf{Z} - \mathcal{Q}\mathbf{Y}$. In this parameterization the linear dilaton is along the $\mathbf{Z}$-direction;

$$\Phi = -\mathrm{Re}\,\mathbf{Z}\,. \tag{3.67}$$

One can directly recover the geometry of $ALE(A_{n-1})$-fibration over $\mathbb{C}P^1$;

$$e^Y + e^{-Y} + X^n + w_1^2 + w_2^2 = 0\,, \tag{3.68}$$

by integrating out the chiral superfield $\mathbf{Z}$ (with the rescaling $\mathbf{X} \to e^{-\mathbf{Z}}\mathbf{X}$) [26].

Similarly, the example **3.1**: $\hat{c} = 4$, $M_{n-2} \otimes L_{3n,1} \otimes L_{3n,1} \otimes L_{3n,1}$ can be identified as the LG theory with

$$W = \mathbf{X}^n + e^{-n\mathbf{Z}}(e^{\mathbf{Y}_1} + e^{\mathbf{Y}_2} + e^{-\mathbf{Y}_1 - \mathbf{Y}_2})\,, \tag{3.69}$$

where we have again the linear dilaton $\Phi = -\mathrm{Re}\,\mathbf{Z}$. This model is shown to describe the $ALE(A_{n-1})$ fibration over $\mathbb{C}P^2$ [26].

Other examples are also identified in a similar manner, leading to the geometrical interpretations mentioned above. For instance, the model **2.2**: $\hat{c} = 3$, $M_{n-2} \otimes L_{n\mu,K_1} \otimes L_{n\mu,K_2}$ $(K_1 + K_2 = \mu)$ is identified with the LG theory with

$$W = \mathbf{X}^n + e^{-n\mathbf{Z}}(e^{\mathbf{Y}/K_1} + e^{-\mathbf{Y}/K_2})\,, \qquad \Phi = -\mathrm{Re}\,\mathbf{Z}\,. \tag{3.70}$$

It corresponds to the non-compact Calabi-Yau geometry

$$e^{Y/K_1} + e^{-Y/K_2} + X^n + w_1^2 + w_2^2 = 0\,, \tag{3.71}$$

which has the structure of $ALE(A_{n-1})$-fibration over $W\mathbb{C}P^1\,[K_1, K_2]$.

[4]Here we absorbed the cosmological constants μ_1, μ_2 by shifting zero-modes of $\mathbf{X}_1$, $\mathbf{X}_2$.

4. D-branes in non-compact models

4.1 Cardy states for compact BPS D-branes

We next study the open string sectors in the non-compact Gepner models. It is well-known that the $\mathcal{N}=2$ superconformal symmetry allows two types of boundary conditions [45];[5]

$$\textbf{A-type}: \quad (J_n - \tilde{J}_{-n})|B\rangle = 0\,, \qquad (G_r^{\pm} - i\tilde{G}_{-r}^{\mp})|B\rangle = 0\,, \tag{4.1}$$

$$\textbf{B-type}: \quad (J_n + \tilde{J}_{-n})|B\rangle = 0\,, \qquad (G_r^{\pm} - i\tilde{G}_{-r}^{\pm})|B\rangle = 0\,, \tag{4.2}$$

which are compatible with the $\mathcal{N}=1$ superconformal symmetry

$$(L_n - \tilde{L}_{-n})|B\rangle = 0\,, \qquad (G_r - i\tilde{G}_{-r})|B\rangle = 0\,, \tag{4.3}$$

where $G = G^+ + G^-$ is the $\mathcal{N}=1$ supercurrent. In the following we shall concentrate on the BPS D-branes with compact world-volumes, which play the fundamental role in the non-compact Calabi-Yau manifolds. Compact branes in L_{N_j,K_j}-sectors are described by the 'class 1' Cardy states in the classification of [1], namely, the ones associated to the extended graviton representations. It can be shown that the consistency with the left-right pairing of the closed string spectrum (3.35) allows *only* the B-type boundary condition for these compact branes [46, 6, 8].

We will begin our analysis by summarizing characteristic aspects of boundary states in each sector.[6] We assume in the following that all the Ishibashi states satisfy the B-type boundary condition.

1. minimal sector M_k**:** (see e.g. [47])

Let $|\ell,m\rangle\rangle^{(\mathrm{NS})}$ ($|\ell,m\rangle\rangle^{(\mathrm{R})}$) be the Ishibashi states in the NS (R) sector characterized by the orthogonality condition ($\sigma=$ NS, R)

$$\begin{aligned}&{}^{(\sigma)}\langle\langle \ell,m|\, e^{-\pi T H^{(c)}} e^{2\pi i z J_0}\,|\ell',m'\rangle\rangle^{(\sigma')} =\\ &= \epsilon_\sigma \delta_{\sigma,\sigma'}\left(\delta_{\ell,\ell'}\delta_{m,m'} + \delta_{\ell,k-\ell'}\delta_{m,m'+k+2}\right)\mathrm{ch}^{(\sigma)}_{\ell,m}(iT,z)\,,\end{aligned} \tag{4.4}$$

where $H^{(c)} = L_0 + \tilde{L}_0 - \frac{c}{12}$ is the closed string hamiltonian and $\mathrm{ch}^{(\mathrm{NS})}_{\ell,m}(\tau,z)$ ($\mathrm{ch}^{(\mathrm{R})}_{\ell,m}(\tau,z)$) denotes the NS (R) character of the $\mathcal{N}=2$ minimal model for the primary field with $h = \dfrac{\ell(\ell+2)-m^2}{4(k+2)}$, $Q = \dfrac{m}{k+2}$ ($h = \dfrac{\ell(\ell+2)-m^2}{4(k+2)} + \dfrac{1}{8}$, $Q = \dfrac{m}{k+2} \pm \dfrac{1}{2}$). (See appendix B.) We have introduced an extra phase factor $\epsilon_\sigma = +1, -1$ for $\sigma = \mathrm{NS}, \mathrm{R}$ respectively for convenience in imposing the GSO projection for supersymmetric D-branes. We also set $|\ell,m\rangle\rangle^{(\mathrm{NS})} = 0$ ($|\ell,m\rangle\rangle^{(\mathrm{R})} = 0$), if $\ell+m \in 2\mathbb{Z}+1$ ($\ell+m\in 2\mathbb{Z}$). The Cardy states are expressed as follows ($\sigma=$ NS, or R , and we set $L+M\in 2\mathbb{Z}$);

$$|L,M\rangle^{(\sigma)} = \sum_{\ell=0}^{k}\sum_{m\in\mathbf{Z}_{2(k+2)}} C_{L,M}(\ell,m)\,|\ell,m\rangle\rangle^{(\sigma)}\,, \qquad C_{L,M}(\ell,m) = \frac{S^{L,M}_{\ell,m}}{\sqrt{S^{0,0}_{\ell,m}}}\,, \tag{4.5}$$

where $S^{\ell,m}_{\ell',m'}$ is the modular coefficients of $\mathrm{ch}^{(\mathrm{NS})}_{\ell,m}(\tau,z)$ (B.12).

[5] A slightly different convention is often used in literature; A-type brane, for instance, is defined by $(G_r^{\pm} - i\eta\tilde{G}^{\mp}_{-r})|B;\eta\rangle = 0$, $(\eta=\pm1)$. The relation to our convention is given by $|B;+1\rangle = |B\rangle$, $|B;-1\rangle = (-1)^{F_R}|B\rangle$.

[6] Here we use notations slightly different from [1].

2. $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$**-sector** $L_{N,K}$**:**

The relevant formulas of boundary states in this sector are given in [1]. The (B-type) Ishibashi states corresponding to continuous and discrete representations are characterized by the relations

$$
\begin{aligned}
{}^{(\sigma)}{}_{\mathbf{c}}\langle\langle p,m|\, e^{-\pi T H^{(c)}} e^{2\pi i z J_0}\, |p',m'\rangle\rangle_{\mathbf{c}}{}^{(\sigma')} &= \epsilon_\sigma \delta_{\sigma,\sigma'}\delta^{(2NK)}_{m,m'}\delta(p-p')\,\chi_{\mathbf{c}}{}^{(\sigma)}(p,m;iT,z)\,,\\
&\quad (p,p'>0)\,, \qquad (4.6)\\
{}^{(\sigma)}{}_{\mathbf{d}}\langle\langle s,m|\, e^{-\pi T H^{(c)}} e^{2\pi i z J_0}\, |s',m'\rangle\rangle_{\mathbf{d}}{}^{(\sigma')} &= \epsilon_\sigma \delta_{\sigma,\sigma'}\delta^{(2NK)}_{m,m'}\delta_{s,s'}\,\chi_{\mathbf{d}}{}^{(\sigma)}(s,m;iT,z)\,, \qquad (4.7)
\end{aligned}
$$

where the range of s is $K+1 \le s \le N+K-1$. We need not introduce Ishibashi states at the boundary $s=K$, $N+K$ as is discussed in [1]. We here set

$$
|s,m\rangle\rangle_{\mathbf{d}}{}^{(\mathrm{NS})} = 0\,,\ \text{unless } s-m\in 2K\mathbb{Z}\,, \qquad |s,m\rangle\rangle_{\mathbf{d}}{}^{(\mathrm{R})} = 0\,,\ \text{unless } s-m\in K(2\mathbb{Z}+1)\,. \tag{4.8}
$$

The Cardy states necessary for our analysis are the class 1 states given in [1] ($R\in\mathbb{Z}_N$);[7]

$$
\begin{aligned}
|R\rangle^{(\sigma)} &= \sum_{s=K+1}^{N+K-1}\sum_{m\in\mathbf{Z}_{2NK}} C_R^{(\sigma)}(s,m)\,|s,m\rangle\rangle_{\mathbf{d}}{}^{(\sigma)} +\\
&\quad + \sum_{m\in\mathbf{Z}_{2NK}}\int_0^\infty dp\,\Psi_R^{(\sigma)}(p,m)\,|p,m\rangle\rangle_{\mathbf{c}}{}^{(\sigma)}\,, \qquad (4.9)\\
C_R^{(\mathrm{NS})}(s,m) &= C_R^{(\mathrm{R})}(s,m) = \left(\frac{2}{N}\right)^{1/2} e^{-2\pi i\frac{Rm}{N}}\sqrt{\sin\left(\frac{\pi(s-K)}{N}\right)}\,, \qquad (4.10)\\
\Psi_R^{(\sigma)}(p,m) &= \left(\frac{2^3K}{N^3}\right)^{1/4} e^{-2\pi i\frac{Rm}{N}} \times\\
&\quad\times\frac{\Gamma\left(\frac{1}{2}+\frac{m-\nu(\sigma)K}{2K}+i\sqrt{\frac{N}{2K}}p\right)\Gamma\left(\frac{1}{2}-\frac{m-\nu(\sigma)K}{2K}+i\sqrt{\frac{N}{2K}}p\right)}{\Gamma\left(i\sqrt{\frac{2K}{N}}p\right)\Gamma(1+i\sqrt{\frac{2N}{K}}p)}\,, \qquad (4.11)
\end{aligned}
$$

where the symbol $\nu(\sigma)$ means $\nu(\mathrm{NS})=0$, $\nu(\mathrm{R})=1$. Cylinder amplitudes of the class 1 Cardy states produce characters of identity representations;

$$
\begin{aligned}
e^{\pi\hat{c}\frac{z^2}{T}}\cdot {}^{(\mathrm{NS})}\langle R|e^{-\pi T H^{(c)}}e^{2\pi i z J_0}|R'\rangle^{(\mathrm{NS})} &= \chi_0^{(\mathrm{NS})}(2K(R'-R);it,z')\,, \qquad (4.12)\\
e^{\pi\hat{c}\frac{z^2}{T}}\cdot {}^{(\mathrm{R})}\langle R|e^{-\pi T H^{(c)}}e^{2\pi i z J_0}|R'\rangle^{(\mathrm{R})} &= \chi_0^{(\widetilde{\mathrm{NS}})}(2K(R'-R);it,z')\,, \qquad (4.13)\\
&\quad\left(T\equiv\frac{1}{t}, z'=-itz\right).
\end{aligned}
$$

The desired Cardy states for our non-compact Gepner model (3.1) should be constructed as

$$
|B;\{L_i,M_i\},\{R_i\};\pm\rangle = |B;\{L_i,M_i\},\{R_i\}\rangle^{(\mathrm{NS})} \pm |B;\{L_i,M_i\},\{R_i\}\rangle^{(\mathrm{R})}\,,
$$

[7] Recall that our closed string Hilbert space for the $L_{N,K}$-piece includes the twisted sectors generated by spectral flows unlike the 'cigar CFT' given in [46, 6, 8]. Note that the D0-brane of cigar CFT corresponds to the boundary state $\sum_{R\in\mathbf{Z}_N}|R\rangle^{(\sigma)}$ in our notation, where the summation over $R\in\mathbb{Z}_N$ eliminates the twisted sectors.

$$|B;\{L_i,M_i\},\{R_i\}\rangle^{(\sigma)} = \mathcal{N}\, P_{\text{closed}}\left[\prod_i |L_i,M_i\rangle^{(\sigma)} \otimes \prod_j |R_j\rangle^{(\sigma)}\right]. \tag{4.14}$$

In the first equation $\pm$ refers to branes and anti-branes, respectively. $\mathcal{N}$ is an overall normalization constant determined by the Cardy condition (its explicit value is not important for our analysis), and P_{closed} means the projection to the closed string Hilbert space determined in our previous analysis. Namely, P_{closed} imposes the following two constraints on the Ishibashi states $|\ell_i,m_i\rangle\rangle^{(\sigma)}$, $|p_j,K_jn_j+N_jw_j\rangle\rangle_{\mathbf{c}}{}^{(\sigma)}$, and $|s_j,K_jn_j+N_jw_j\rangle\rangle_{\mathbf{d}}{}^{(\sigma)}$;

- Integrality of the total U(1)-charge in the NS-sector;

$$\sum_i \frac{m_i}{k_i+2} + \sum_j \frac{K_jn_j}{N_j} \in \mathbb{Z}\,. \tag{4.15}$$

- The consistency with the B-type boundary condition, which gives essentially the same condition as (3.39);

$$^{\exists}r \in \mathbb{Z}_N \;\; \text{s.t. } m_i \equiv -r \pmod{k_i+2}\,, \quad n_j \equiv r \pmod{N_j}\,, \quad (^{\forall}i, {}^{\forall}j)\,. \tag{4.16}$$

(4.16) is derived as follows; due to the left-right pairing in our construction (3.35) a typical state in the closed string spectrum has the form

$$\begin{aligned} &|\ell_i,m_i\rangle \otimes |\tilde{\ell}_i=\ell_i,\tilde{m}_i=m_i\rangle : \qquad\qquad \text{in the } M_{k_i}\text{-sector,}\\ &|s_j,m_j=K_jn_j+N_jw_j\rangle \otimes |\tilde{s_j}=N_j+2K_j-s_j,\tilde{m_j}=K_jn_j-N_jw_j+N_j\rangle : \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in the } L_{N_j,K_j}\text{-sector}\,. \end{aligned} \tag{4.17}$$

Corresponding B-type boundary state has the quantum numbers as

$$\begin{aligned} &|\ell_i,m_i\rangle \otimes |\tilde{\ell}_i=\ell_i,\tilde{m}_i=-m_i\rangle,\\ &|s_j,m_j=K_jn_j+N_jw_j\rangle \otimes |\tilde{s_j}=N_j+2K_j-s_j,\tilde{m_j}=-K_jn_j-N_jw_j+N_j\rangle. \end{aligned} \tag{4.18}$$

(4.18) can be obtained from (4.17) by spectral flow in the right moving sector if (4.16) is obeyed. Analysis of massive Ishibashi states is similar.

Because of the second constraint (4.16), the Cardy states actually depends only on the sum of labels

$$M \equiv \sum_i \mu_i M_i + 2\sum_j \nu_j K_j R_j \;\in \mathbb{Z}_{2N}\,, \tag{4.19}$$

and thus we shall write them as $|B;\{L_i\},M;\pm\rangle$, $|B;\{L_i\},M\rangle^{(\sigma)}$ from here on.

It is now possible to count the numbers of compact D-branes and compare them with those of massless states in the models with $N_M=1$, $1\le N_L\le 3$. See table 1. We note that some of the D-branes (cycles) do not have corresponding massless states and thus the number of D-brane exceed those of massless moduli. This is because the "would be" moduli which are non-normalizable in the non-compact geometry do not appear as massless states in the closed string spectrum.

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Models	Geometric Identification	No. of massless states	No. of basic vanishing cycles
$M_{n-2} \otimes L_{n,1}$	$ALE\ (A_{n-1})$	$n-1$	n
$M_{n-2} \otimes L'_{2n,n+2}$	$CY_3\ (A_{n-1})$	$\left[\frac{n-2}{2}\right]+1$	n
$M_{n-2} \otimes L_{n,n+1}$	$CY_4\ (A_{n-1})$	0	n
$M_{n-2} \otimes L_{2n,1} \otimes L_{2n,1}$	$ALE\ (A_{n-1})$ fibration over $\mathbb{C}P^1$	$n-1$	$2n$
$M_{n-2} \otimes L_{n\mu,K_1} \otimes L_{n\mu,K_2}$, $K_1+K_2=\mu$, G.C.D$\{K_i\}=1$	$ALE\ (A_{n-1})$ fibration over $W\mathbb{C}P^1[K_1,K_2]$	$n-1$	μn
$M_{n-2} \otimes L'_{4n,n+2} \otimes L'_{4n,n+2}$	$CY_3\ (A_{n-1})$ fibration over $\mathbb{C}P^1$	$\left[\frac{n-2}{2}\right]+1$	$2n$
$M_{n-2} \otimes L'_{2n\mu,K_1(n+2)} \otimes L'_{2n\mu,K_2(n+2)}$, $K_1+K_2=\mu$, G.C.D$\{K_i\}=1$	$CY_3\ (A_{n-1})$ fibration over $W\mathbb{C}P^1[K_1,K_2]$	$\left[\frac{n-2}{2}\right]+1$	μn
$M_{n-2} \otimes L_{3n,1} \otimes L_{3n,1} \otimes L_{3n,1}$	$ALE\ (A_{n-1})$ fibration over $\mathbb{C}P^2$	$n-1$	$3n$
$M_{n-2} \otimes L_{n\mu,K_1} \otimes L_{n\mu,K_2} \otimes L_{n\mu,K_3}$, $K_1+K_2+K_3=\mu$, G.C.D$\{K_i\}=1$	$ALE\ (A_{n-1})$ fibration over $W\mathbb{C}P^2[K_1,K_2,K_3]$	$n-1$	μn

Table 1: Spectra of compact BPS branes and closed string massless states. ("No. of basic vanishing cycles" means the number of compact BPS branes $|B;L,M\rangle$ with $L=0$. They are not necessarily homologically independent. Generic cycles with $L\neq 0$ are expressed as superpositions of the basic ones.)

For the sake of later use let us derive the 'charge integrality' condition for the R-sector. This is obtained from (4.15) by a 1/2-spectral flow: first, U(1)-charges of M_{K_i} and L_{N_j,K_j}-sectors are shifted under the flow as

$$\frac{m_i}{k_i+2} \Longrightarrow \frac{1}{2}+\frac{m'_i}{k_i+2}\,, \qquad (m'_i \equiv m_i-1) \tag{4.20}$$

$$\frac{K_j n_j}{N_j} \Longrightarrow \frac{1}{2}+\frac{K_j n'_j}{N_j}\,, \qquad (n'_j \equiv n_j+1)\,. \tag{4.21}$$

At the same time total U(1)-charge gets shifted by $\hat{c}/2$. Thus the condition for charge integrality in R-sector becomes

$$\frac{m'_i}{k_i+2}+\frac{K_j n'_j}{N_j} \in \mathbb{Z}+\frac{\gamma}{2}\,, \tag{4.22}$$

where γ is defined as

$$\gamma = \hat{c}-(N_M+N_L)\,. \tag{4.23}$$

As it turns out, our models have different characteristics depending on the even-oddness of the parameter γ.

A comment on the $\hat{c}=2$ case: since we showed that the (c,c) and (a,a)-type chiral primaries exist in the $\hat{c}=2$ case, one may suppose that the A-type compact branes also exist in $\hat{c}=2$ theory. However, this is not the case. In fact, the (a,a)-type chiral primary states (3.41) contains in each M_{k_i}-sector a state of the type

$$(\ell_i,-\ell_i)_L \otimes (k_i-\ell_i,\ell_i-k_i)_R\,. \tag{4.24}$$

Thus we cannot define the A-type Ishibashi state except for the special case $\ell_i = k_i/2$. Therefore, generic $\hat{c}=2$ theories do not have sufficient number of A-type Ishibashi states for constructing compact A-branes.

Strictly speaking, there is an exception; $L_{2,1}$ (with no M_k factor), which corresponds to the Eguchi-Hanson space topologically equivalent with T^*S^2. In this case we can construct the A-type boundary state $|B;O\rangle_A$ for compact brane as well as the B-type $|B;O\rangle_B$, both of which are associated to the $\mathcal{N}=4$ massless character of $\ell=0$ [48]. This fact seems to contradict with the geometrical interpretation, since only one cycle $\cong S^2$ exists in the Eguchi-Hanson space. However, this apparent puzzle is resolved by analyzing cylinder amplitudes. The B-B and A-A overlaps become (in the NS sector)

$$
{}_{\mathrm{B}}\langle B;O|e^{-\pi T H^{(c)}}|B;O\rangle_{\mathrm{B}} = {}_{\mathrm{A}}\langle B;O|e^{-\pi T H^{(c)}}|B;O\rangle_{\mathrm{A}} = \mathrm{ch}_0^{\mathcal{N}=4}(\ell=0;it,0)\,,\quad \left(T\equiv\frac{1}{t}\right) \tag{4.25}
$$

as expected. Here $\mathrm{ch}_0^{\mathcal{N}=4}(\ell=0;\tau,z)$ is the $\mathcal{N}=4$ massless character of $\ell=0$ [48]. On the other hand, the A-B overlap is evaluated as follows;

$$
{}_{\mathrm{A}}\langle B;O|e^{-\pi T H^{(c)}}|B;O\rangle_{\mathrm{B}} = \chi_{(-,+)}(p=i/2;it) - \int_0^\infty dp\,\frac{2}{\cosh\pi p}\,\chi_{(-,+)}(p;it)\,,\quad \left(t\equiv\frac{1}{T}\right), \tag{4.26}
$$

where $\chi_{(-,+)}(p;\tau)$ is the twisted $\mathcal{N}=2$ character defined in (D.4). Twisted character appears due to the difference in the boundary conditions. We have used the fact that the absolute value squared of boundary wave function becomes

$$
2\sinh\pi p'\tanh\pi p' = 2\left(\cosh\pi p' - \frac{1}{\cosh\pi p'}\right), \tag{4.27}
$$

and also used a contour deformation technique, which yields the first term in (4.26). Note that the second term in (4.26) appears with a negative sign, which means that the open channel amplitude includes negative norm states. This implies that compact A and B-branes are mutually incompatible, and we discard the A-brane as in the other cases. In conclusion our brane spectrum matches with the geometrical expectation also in this case.

4.2 Cylinder amplitudes

Now, let us analyze cylinder amplitudes ending on the compact BPS branes in various models. We assume the parameter γ (4.23) to be an even integer $\gamma\in 2\mathbb{Z}$ for the time being. Calculation of various amplitudes becomes simplified under this assumption.

We start our analysis by working on the NS-sector amplitudes;

$$
Z^{(\mathrm{NS})}(\{L_i\},M|\{L_i'\},M')(it) \equiv {}^{(\mathrm{NS})}\langle B;\{L_i\},M|e^{-\pi T H^{(c)}}|B;\{L_i'\},M'\rangle^{(\mathrm{NS})},\quad \left(T\equiv\frac{1}{t}\right). \tag{4.28}
$$

The calculation is quite similar to the cylinder amplitudes for the B-branes in the compact Gepner models (see e.g. [47, 49]). The non-trivial point is the treatment of the projection operator P_{closed}. The following formulas are useful in imposing the second constraint (4.16);

$$
\sum_{\substack{a\in\mathbf{Z}_{2(k+2)}\\ L+a\in 2\mathbf{Z}}} e^{-2\pi i\frac{am}{2(k+2)}}\,\mathrm{ch}_{L,a}^{(\mathrm{NS})}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \tag{4.29}
$$

$$
= e^{i\pi\frac{k}{k+2}\frac{z^2}{\tau}}\sum_{\ell=0}^{k}\sin\left(\frac{\pi(L+1)(\ell+1)}{k+2}\right)\left[\mathrm{ch}_{\ell,m}^{(\mathrm{NS})}(\tau,z)+(-1)^L\mathrm{ch}_{\ell,m+k+2}^{(\mathrm{NS})}(\tau,z)\right]
$$

$$
\begin{aligned}
&\sum_{R\in\mathbf{Z}_N} e^{2\pi i \frac{RKn}{N}}\, \chi_0^{(\mathrm{NS})}\left(2KR; -\frac{1}{\tau}, \frac{z}{\tau}\right) = \\
&= e^{i\pi \hat{c}_L \frac{z^2}{\tau}} \sqrt{\frac{N}{2K}} \sum_{w\in\mathbf{Z}_{2K}} \left[\int_0^\infty dp\, \frac{\sinh\left(\pi\sqrt{\frac{2K}{N}}p\right)\sinh\left(\pi\sqrt{\frac{2N}{K}}p\right)}{\left|\cosh\pi\left(\sqrt{\frac{N}{2K}}p + i\frac{Kn+Nw}{2K}\right)\right|^2} \chi_{\mathbf{c}}{}^{(\mathrm{NS})}\times \right. \\
&\qquad\qquad \times (p, Kn+Nw;\tau,z) + \sum_{s=K+1}^{N+K-1} 2\sin\left(\frac{\pi(s-K)}{N}\right)\times \\
&\qquad\qquad \left. \times \chi_{\mathbf{d}}{}^{(\mathrm{NS})}(s, Kn+Nw;\tau,z)\right], \quad \left(\hat{c}_L = 1+\frac{2K}{N}\right). \qquad (4.30)
\end{aligned}
$$

We obtain (up to an overall normalization)

$$
\begin{aligned}
Z^{(\mathrm{NS})}(\{L_i\}, M|\{L'_i\}, M')(it) &\propto \sum_{\ell_i} \sum_{\substack{a_i\in\mathbf{Z}_{2(k_i+2)}\\ a_i\equiv\ell_i\,(\mathrm{mod}\,2)}} \sum_{a'_j\in\mathbf{Z}_{N_j}} \sum_{a\in\mathbf{Z}_N} \sum_{r\in\mathbf{Z}_{2N}} \frac{1}{N}\cdot\frac{1}{2N}\times \\
&\quad \times \exp\left[2\pi i \frac{r}{2N}\left\{M'-M+\sum_i \mu_i a_i + 2\sum_j \nu_j K_j a'_j\right\}\right]\times \\
&\quad \times \prod_i\prod_j N^{\ell_i}_{L_i,L'_i}\, \mathrm{ch}^{(\mathrm{NS})}_{\ell_i,a_i+2a}(it,0)\, \chi_0^{(\mathrm{NS})}(2K_j(a'_j-a);it,0) \\
&= \sum_{\ell_i} \sum_{\substack{a_i\in\mathbf{Z}_{2(k_i+2)}\\ a_i\equiv\ell_i\,(\mathrm{mod}\,2)}} \sum_{a'_j\in\mathbf{Z}_{N_j}} \sum_{a\in\mathbf{Z}_N} \frac{1}{N}\,\delta^{(2N)}\left(M'-M+\sum_i\mu_i a_i + 2\sum_j \nu_j K_j a'_j\right)\times \\
&\quad \times \prod_i\prod_j N^{\ell_i}_{L_i,L'_i}\, \mathrm{ch}^{(\mathrm{NS})}_{\ell_i,a_i+2a}(it,0)\, \chi_0^{(\mathrm{NS})}(2K_j(a'_j-a);it,0)\,. \qquad (4.31)
\end{aligned}
$$

Here $N^{\ell_i}_{L_i,L'_i}$ are the fusion coefficients of $\mathrm{SU}(2)_{k_i}$;

$$
N^{\ell_i}_{L_i,L'_i} = \begin{cases} 1 & |L_i-L'_i|\le\ell_i\le\min[L_i+L'_i,\, 2k_i-L_i-L'_i] \text{ and } \ell_i\equiv|L_i-L'_i|\ (\mathrm{mod}\,2) \\ 0 & \text{otherwise}\end{cases}. \qquad (4.32)
$$

The integrality of total U(1)-charge is ensured by the a-summation, while the a_i and a'_j summations impose the constraint (4.16) via the relations (4.29), (4.30). Thanks to our assumption

$$
2\sum_i \mu_i - 2\sum_j \nu_j K_j = -N\gamma \in 2N\mathbb{Z}\,, \qquad (4.33)
$$

we may make the shifts $a_i \to a_i+2a$, $a'_j \to a'_j-a$ in the factor $\delta^{(2N)}(\cdots)$. The a-summation is then decoupled and we obtain a simpler form of amplitude

$$
Z^{(\mathrm{NS})}(\{L_i\}, M|\{L'_i\}, M')(it) \propto
$$

$$\propto \sum_{\ell_i} \sum_{\substack{a_i \in \mathbf{Z}_{2(k_i+2)} \\ a_i \equiv \ell_i \,(\mathrm{mod}\, 2)}} \sum_{a'_j \in \mathbf{Z}_{N_j}} \delta^{(2N)} \left(M' - M + \sum_i \mu_i a_i + 2 \sum_j \nu_j K_j a'_j \right) \times$$
$$\times \prod_i \prod_j N^{\ell_i}_{L_i, L'_i} \, \mathrm{ch}^{(\mathrm{NS})}_{\ell_i, a_i}(it, 0) \, \chi_0^{(\mathrm{NS})}(2K_j a'_j; it, 0) \,. \tag{4.34}$$

In the cases of $\gamma \in 2\mathbb{Z}+1$ the a-summation is in general not decoupled and the calculation becomes more complex.

We next consider the open string Witten indices defined by

$$I(\{L_i\}, M | \{L'_i\}, M') = {}^{(\mathrm{R})}\langle B; \{L_i\}, M | e^{-\pi T H^{(c)}} e^{-i\pi J_0} | B; \{L'_i\}, M' \rangle^{(\mathrm{R})} \,. \tag{4.35}$$

Under the assumption $\gamma \in 2\mathbb{Z}$, the U(1)-charge condition for R-sector has the same form as (4.15). In order to evaluate the amplitudes we only have to replace the NS-characters $\mathrm{ch}^{(\mathrm{NS})}_{*,*}(it, 0)$, $\chi_0^{(\mathrm{NS})}(*; it, 0)$ by the $\widetilde{\mathrm{R}}$-characters, that is, the Witten indices (up to signs)

$$\begin{aligned} \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell_i, m_i}(it; 0) &= \delta^{(2(k_i+2))}(m_i - (\ell_i + 1)) - \delta^{(2(k_i+2))}(m_i + (\ell_i + 1)) \,, \\ \chi_0^{(\widetilde{\mathrm{R}})}(2K_j r_j; it, 0) &= \delta^{(N_j)} \left(r_j - \frac{1}{2} \right) - \delta^{(N_j)} \left(r_j + \frac{1}{2} \right) , \end{aligned} \tag{4.36}$$

and also replace the summation $\displaystyle\sum_{a \in \mathbf{Z}_N} *$ with $\displaystyle\sum_{a \in \frac{1}{2} + \mathbf{Z}_N} *$ because of the insertion of $e^{-i\pi J_0}$.

We finally obtain

$$\begin{aligned} & I(\{L_i\}, M | \{L'_i\}, M') \propto \\ & \quad \propto \sum_{\ell_i} \sum_{\alpha_i = \pm 1} \sum_{\beta_j = \pm 1} \delta^{(2N)} \left(M' - M + \sum_i \mu_i \alpha_i (\ell_i + 1) + \sum_j \nu_j K_j \beta_j + \frac{N}{2} \gamma \right) \times \\ & \quad \times \prod_i \prod_j N^{\ell_i}_{L_i, L'_i} \, \mathrm{sgn}(\alpha_i) \mathrm{sgn}(\beta_j) \,. \end{aligned} \tag{4.37}$$

The sector in which all of L_i and L'_i equal 0 play the basic role. Following [49], $[I_0]_{M,M'} \equiv I(\{L_i = 0\}, M | \{L'_i = 0\}, M')$ can be concisely expressed by introducing a cyclic operator g defined by the action

$$g \; : \; M \longmapsto M + 2 \,, \tag{4.38}$$

which satisfies $g^N = 1$. Then we obtain from (4.37)

$$\begin{aligned} I_0 &= \prod_i \prod_j \left(g^{\frac{\mu_i}{2}} - g^{-\frac{\mu_i}{2}} \right) \left(g^{\frac{\nu_j K_j}{2}} - g^{-\frac{\nu_j K_j}{2}} \right) g^{\frac{N}{4}\gamma} \\ &\propto \prod_i \prod_j (1 - g^{-\mu_i})(1 - g^{\nu_j K_j}) \,, \qquad (\text{up to overall sign}) \,. \end{aligned} \tag{4.39}$$

In this way we can derive the formula conjectured in [23], which generalizes the one for the B-branes in the compact Gepner models [47, 49]. We here emphasize that the formula (4.39) is correct only in the case with even γ. It is easy to check that, under the assumption

$\gamma \in 2\mathbb{Z}$, the formula (4.39) has the correct symmetry, namely,[8]

$$\begin{aligned} I_0^t &= I_0\,, \qquad \text{for} \qquad \hat{c} = 2,4\,, \\ I_0^t &= -I_0\,, \qquad \text{for} \qquad \hat{c} = 3\,. \end{aligned} \tag{4.40}$$

Let us next discuss the odd γ cases that are somewhat more complicated. We here present two examples.

1. CY_3 (A_{n-1})**:**

In this case we have $\gamma = 3 - 2 = 1$. As was already discussed, we adopt half-integral values of the winding the number

$$w_1 \in \frac{1}{2}\mathbb{Z}_{4(n+2)}\,, \qquad n_1 + 2w_1 \in 2\mathbb{Z}\,. \tag{4.41}$$

It is convenient to parameterize

$$\begin{aligned} &k_1 + 2 = N_1 = N = n\,, \qquad K_1 = \frac{n+2}{2}\,, \qquad (\text{for even } n)\,, \\ &k_1 + 2 = n\,, \qquad N_1 = N = 2n\,, \qquad K_1 = n+2\,, \qquad (\text{for odd } n)\,, \end{aligned} \tag{4.42}$$

so that K_1 and $N_1/2$ (not N_1) are relatively prime.

We first consider the even n case. An important difference from the previous analysis is in the U(1)-charge condition for R-sector;

$$\frac{m}{n} + \frac{(n+2)n_1 + 2nw_1}{2n} \in \mathbb{Z} + \frac{\gamma}{2} = \mathbb{Z} + \frac{1}{2}\,, \tag{4.43}$$

which leads to an extra insertion of $e^{i\pi a}$ in the amplitude. Secondly, the "$w \in \frac{1}{2}\mathbb{Z}$-rule" (4.41) gives rise to a replacement $a_1' \to 2a_1'$. We so obtain

$$\begin{aligned} &I(L,M|L',M') \propto \\ &\propto \sum_{\ell} \sum_{\substack{a_1 \in \mathbf{Z}_{2n} \\ a_1 \equiv \ell \,(\mathrm{mod}\, 2)}} \sum_{a_1' \in \mathbf{Z}_n} \sum_{a \in \mathbf{Z}_n} \sum_{r \in \mathbf{Z}_{2n}} \frac{1}{n} \cdot \frac{1}{2n} \exp\left[2\pi i \frac{r}{2n}\left\{M' - M + a_1 + (n+2)2a_1'\right\}\right] \times \\ &\quad \times e^{i\pi a}\, N_{L,L'}^{\ell}\, \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{\ell, a_1 + 2a}(it,0)\, \chi_0^{(\widetilde{\mathrm{R}})}((n+2)(2a_1' - a); it, 0)\,. \end{aligned} \tag{4.44}$$

Since only the terms with $2a_1' - a = \pm\frac{1}{2}$ contribute, we may replace the factor $e^{i\pi a}$ with $e^{-i\frac{\pi}{2}\beta}$ ($\frac{\beta}{2} \equiv 2a_1' - a$, $\beta = \pm 1$), which cancels out the factor $\mathrm{sgn}(\beta)$ in (4.37). The summation over a is again decoupled. We finally obtain the formula

$$I(L,M|L',M') \propto \sum_{\ell} \sum_{\alpha = \pm 1} \sum_{\beta = \pm 1} \delta^{(2n)}\left(M' - M + \alpha(\ell+1) + \beta\right)\, N_{L,L'}^{\ell}\, \mathrm{sgn}(\alpha)\,. \tag{4.45}$$

In the $L = 0$ sector, we have

$$I_0 \propto (1 - g^{-1})(1 + g) = g - g^{-1}\,, \tag{4.46}$$

that is anti-symmetric as is expected.

[8] One way to confirm the result (4.40) is to take the T-dual so that L_{N_j,K_j}-sectors are realized as $\mathcal{N}=2$ Liouville theories. There, the compact branes are A-branes corresponding to middle dimensional cycles.

In the odd n case the story becomes more complicated. Since the label of the Cardy states R_1 in $L'_{2n,n+2}$-sector runs over the range $R_1 \in \mathbb{Z}_{2n}$, it appears that there exist twice as many BPS branes. However, the proper Cardy states in $L'_{2n,n+2}$ have to be compatible with (4.41) and are given as $|R_1\rangle^{(\mathrm{R})} + |R_1+n\rangle^{(\mathrm{R})}$. We thus still have the same number of compact branes, and one may restrict the label M to the range $M \in \mathbb{Z}_{4n} \cap 2\mathbb{Z}$. Rewriting $M/2$ as M, we can obtain the same formula of Witten indices (4.45) (and (4.46)).

2. $\hat{c}=4$**,** $M_{n-2} \otimes L'_{4n,n+2} \otimes L'_{4n,n+2}$**:**

This model has been identified as the $CY_3(A_{n-1})$-fibration over $\mathbb{C}P^1$ and we have $\gamma = 4-3 = 1$. We should again apply the "$w \in \frac{1}{4}\mathbb{Z}$ rule" (3.15) to the $L'_{4n,n+2}$-sectors, and make the replacements $a'_i \to 4a'_i$. A similar calculation leads to

$$I(L,M|L',M') \propto \sum_{\ell} \sum_{\alpha=\pm 1} \sum_{\beta_1=\pm 1} \sum_{\beta_2=\pm 1} \times \\ \times \delta^{(4n)}\left(M'-M+2\alpha(\ell+1)+\beta_1+\beta_2\right) N^{\ell}_{L,L'}\, \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta_1)\,, \tag{4.47}$$

and also in the $L=0$ sector,

$$I_0 \propto (1-g^{-2})(1-g)(1+g) = 2 - g^2 - g^{-2}\,. \tag{4.48}$$

Note that the factor $\mathrm{sgn}(\beta_2)$ was canceled by $e^{i\pi a}$ in the same way as the first example, yielding a contribution $1+g$ in place of $1-g$ in (4.48). This fact makes (4.48) symmetric as should be.

4.3 Comments on non-compact BPS branes

We can similarly investigate aspects of non-compact BPS branes. The analysis is straightforward but more cumbersome technically. We thus restrict ourselves to making some comments about non-trivial points. We shall concentrate on the non-compact branes associated to the massive representations in each of L_{N_j,K_j}-sectors ('class 2' in the classification in [1]), and focus on the NS-sector.

The first fact which is in contrast to the case of compact branes is that *both* the A and B-type boundary conditions are possible. This is because the U(1)-charges of massive representations are uncorrelated with their conformal weights. The desired Cardy states describing non-compact branes are now constructed in a similar manner to (4.14): all we have to do is to take the products of Cardy states in each sector and to let the projection P_{closed} act on them.

The Cardy states in each of the M_{k_i}-sectors are given as in (4.5) (here we specify the A and B-type boundary conditions by subscripts);

$$|L_i, M_i\rangle_A = \sum_{\ell_i} \sum_{m_i} e^{i\pi\frac{\ell_i}{2}} C_{L_i,M_i}(\ell_i,m_i)\, |\ell_i,m_i\rangle\rangle_A\,, \tag{4.49}$$

$$|L_i, M_i\rangle_B = \sum_{\ell_i} \sum_{m_i} C_{L_i,M_i}(\ell_i,m_i)\, |\ell_i,m_i\rangle\rangle_B\,. \tag{4.50}$$

Note that the A-B overlap of Ishibashi states yields the twisted minimal characters (see appendix D), and the extra phase factor $e^{i\pi\frac{\ell_j}{2}}$ in (4.49) is necessary for the consistency with the modular bootstrap. (See the modular transformation formulas (D.8), and recall the fact that the identity brane $|B;O\rangle$ is a B-brane.)

The Cardy states in the L_{N_j,K_j}-sectors are more non-trivial. Since the branes are non-compact, we should allow the continuous spectrum of U(1)-charge in the open channel, while that in the closed channel should be still discrete. We thus have to introduce a one-parameter deformation of the extended massive characters (C.16)

$$\chi_{\mathbf{c}}{}^{(\mathrm{NS})}(p,m;\tau,z,w) = q^{\frac{p^2}{2}}\Theta_{m,NK}\left(\tau,\frac{2z}{N}-\frac{w}{NK}\right)\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,,\quad (w\in\mathbb{R},\quad 0\leq w<2NK)\,, \tag{4.51}$$

and the associated Ishibashi states $|p,m,\alpha\rangle\rangle_A$, $|p,m,\alpha\rangle\rangle_B$ ($\alpha\in\mathbb{R}$, $0\leq\alpha<2NK$) by the relations;

$$\begin{aligned}
{}_A\langle\langle p,m,\alpha|\, e^{-\pi T H^{(c)}}e^{2\pi i z J_0}\,|p',m',\alpha'\rangle\rangle_A &= {}_B\langle\langle p,m,\alpha|\, e^{-\pi T H^{(c)}}e^{2\pi i z J_0}\,|p',m',\alpha'\rangle\rangle_B \\
&= \delta(p-p')\delta^{(2NK)}_{m,m'}\chi_{\mathbf{c}}{}^{(\mathrm{NS})}(p,m;\tau,z,\alpha'-\alpha)\,, \\
&\quad (p,p'>0) \\
{}_A\langle\langle p,m,\alpha|\, e^{-\pi T H^{(c)}}e^{2\pi i z J_0}\,|p',m',\alpha'\rangle\rangle_B &= {}_B\langle\langle p,m,\alpha|\, e^{-\pi T H^{(c)}}e^{2\pi i z J_0}\,|p',m',\alpha'\rangle\rangle_A \\
&= \delta(p-p')\delta^{(2NK)}_{m,0}\delta^{(2NK)}_{m',0}\,\chi_{(+,-)}(p;iT)\,, \\
&\quad (p,p'>0)
\end{aligned} \tag{4.52}$$

where $\chi_{(+,-)}(p;\tau)$ is the twisted massive character defined in (D.4). By these definitions the Ishibashi states $|p,m,\alpha\rangle\rangle_A$, $|p,m,\alpha\rangle\rangle_B$ are explicitly constructed as the spectral flow sums of irreducible Ishibashi states, and the parameter α expresses the relative phase attached to each irreducible one.

The desired pieces of A and B-type Cardy states in the L_{N_j,K_j}-sector are now given as follows [46, 1, 2, 6, 8]: they have different boundary wave functions;

$$|P_j,\alpha_j\rangle_B = \sqrt{\frac{2}{N_jK_j}}\int_0^\infty dp\sum_m \cos(2\pi P_j p)f(p,m)\;|p,m,\alpha_j\rangle\rangle_B\,, \tag{4.53}$$

$$|P_j,\alpha_j\rangle_A = \frac{1}{\sqrt{2N_jK_j}}\int_0^\infty dp\sum_m\left(e^{2\pi i P_j p}+(-1)^m e^{-2\pi i P_j p}\right)f(p,m)\;|p,m,\alpha_j\rangle\rangle_A\,, \tag{4.54}$$

where we set

$$f(p,m)\equiv\frac{1}{\Psi_O^{(\mathrm{NS})}(-p,m)} = \left(\frac{N^3}{2^3K}\right)^{\frac{1}{4}}\frac{\Gamma\left(-i\sqrt{\frac{2K}{N}}p\right)\Gamma\left(1-i\sqrt{\frac{2N}{K}}p\right)}{\Gamma\left(\frac{1}{2}+\frac{m}{2K}-i\sqrt{\frac{N}{2K}}p\right)\Gamma\left(\frac{1}{2}-\frac{m}{2K}-i\sqrt{\frac{N}{2K}}p\right)}\,, \tag{4.55}$$

We present a few comments:

1. The B-type Cardy state (4.53) is determined from the modular bootstrap, while the A-type (4.54) is not. This is because the identity brane is now a B-brane and

hence the modular bootstrap is not powerful to determine the A-type Cardy state. Nevertheless, (4.54), which is called 'class 2′' in [8], has been constructed in [46, 6] as the 'descent' of the AdS_2-brane [50] in the euclidean AdS_3 [51].[9] One can check that the difference of coefficients in (4.53) and (4.54) is consistent with reflection amplitudes of the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$-coset model [52, 14].

In the cigar models (4.53) would correspond to non-compact D2-branes (partially wrapping on cigar) [8], while (4.54) do to non-compact D1-branes [46, 6]. The continuous parameter α_j would express the angular positions and Wilson lines of D1, D2-branes respectively. However, one should keep it in mind that the backgrounds here have different geometries from the cigar due to the orbifolding procedure, and thus the classical DBI analysis on the cigar as in [46] cannot be simply applied to our case.

2. Computations of cylinder amplitudes are straightforward but more complicated than the compact brane case. In the simplest example $M_{N-2}\otimes L_{N,1}$ some analysis has been already done in [9].[10] It is important that all the open string amplitudes appearing in the A-A or B-B type overlaps are expanded by the products of minimal characters and extended massive characters with the *continuous* U(1)-charges ${\chi_{\mathbf{c}}}^{(*)}(p_j, \omega_j; it, 0)$, $\omega_j \in \mathbb{R}$, $0 \leq \omega_j < 2N_jK_j$. (See the modular transformation formula (C.15).)

   For the B-branes the projection P_{closed} acts in the same way as in the compact branes, namely imposes (4.15) and (4.16), and hence the B-branes can depend only on one parameter along the U(1)-direction. As for the A-branes, on the other hand, P_{closed} only imposes the U(1)-charge integrality (4.15) and not the second constraint (4.16). If $K_j > 1$, P_{closed} further gives rise to the summation over the shifts $\omega_j \rightarrow \omega_j + 2N_jl_j$ ($l_j \in \mathbb{Z}_{K_j}$) in the open channel character ${\chi_{\mathbf{c}}}^{(*)}(p_j, \omega_j; it, 0)$, since the A-type Ishibashi states $|p_j, m_j\rangle\rangle_{\mathbf{c}A}$ exist only for $m_j = K_jn_j$ ($n_j \in \mathbb{Z}_{2N_j}$).

   The A-B (or B-A)-type overlaps are expressed by the products of the twisted $\mathcal{N}=2$ characters $\chi_{(-,+)}(p;it)$ (D.4) and $\chi_{L\,(-,+)}(it)$ (D.5). We do not have spectral flow sums in the open channel in this case, since only the U(1)-neutral states contribute to the twisted characters. The Cardy condition is expected to be satisfied with suitable spectral densities (after subtracting the IR divergences) in all these cases.

3. We still have a possibility to construct other types of non-compact BPS branes based on the 'class 3' boundary states [1], which are associated with the massless matter representations in the L_{N_j,K_j}-sectors. In the cigar models it has been pointed out [53, 8] that they could describe the D2-branes found in [46, 6] covering the whole cigar. However, as is suggested from a detailed analysis of cylinder amplitudes performed in [8], it seems difficult that these class 3 branes can satisfy the Cardy condition, as far as we insist on *unitary representations* in the open string channel. We leave this subtle problem to future works.

[9] In the recent paper [10] the boundary wave function of A-type (4.54) has been also derived by the boundary bootstrap approach based on the perturbative analysis in the dual $\mathcal{N}=2$ Liouville theory.

[10] The analysis in [9] corresponds to the case with discrete α.

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4.4 Non-BPS branes

To close this section we discuss the Cardy states for non-BPS D-branes that exhibit some interesting properties. To avoid unessential complexity we shall concentrate on the simplest case of N NS5 branes (or $ALE(A_{N-1})$) described by $M_{N-2} \otimes L_{N,1}$, and only consider the compact branes. Extensions to more general models and the cases of non-compact branes should be straightforward.

The simplest non-BPS branes are of course obtained in the same way as in the flat backgrounds (see e.g. [54]), that is, by projecting out the RR-components of boundary states and by multiplying the remaining NSNS-components by $\sqrt{2}$. These branes are constructed using the descent relation and the $\mathbb{Z}_2$-orbifolding acting by the space-time fermion number. Since we now possess the $\mathbb{Z}_N$-symmetries in the U(1)-charge sector, we may construct more non-trivial non-BPS branes by using the $\mathbb{Z}_N$-orbifolding procedure. This type of non-BPS branes may be regarded as the natural extension of the "unstable B-branes" in the SU(2)-WZW model presented in [31].

The basic prescription for their construction is summarized as follows (we focus on the NSNS-sector for the time being);

1. Start with the compact BPS brane (4.14)

$$|B;L,M\rangle_B = \mathcal{N} P_{\text{closed}} \left[|L,M\rangle_{M_{N-2}} \otimes |R=0\rangle_{L_{N,1}} \right], \qquad (L+M\in 2\mathbb{Z}), \tag{4.56}$$

   which includes the B-type Cardy states for both sectors. Consider the "wrong-dimensional" BPS branes $|B;L,M\rangle'_{AB}$ which is defined by reversing formally the boundary condition in the M_{N-2}-sector of (4.56). The subscript AB indicates to take the A and B-type boundary conditions for the M_{N-2} and $L_{N,1}$-sectors, respectively.

2. Sum up $|B;L,M\rangle'_{AB}$ over the spectral flows in the open string channel, namely, we define

$$|B;L\rangle_{AB} = \sum_{r\in\mathbf{Z}_N} |B;L,L+2r\rangle'_{AB}, \tag{4.57}$$

   which should be the desired boundary states of non-BPS branes.

It is important to note that, although the 'wrong BPS brane' $|B;L,M\rangle'_{AB}$ is not compatible with the charge-integrality condition, (4.57) is consistent because the sum over $r\in\mathbb{Z}_N$ projects out states with fractional U(1)-charges. They actually consist of Ishibashi states with integral U(1)-charges *separately* in each sector, M_{N-2} and $L_{N,1}$. They do not satisfy the $\mathcal{N}=2$ boundary conditions (4.1) or (4.2), and at most preserve the $\mathcal{N}=1$ superconformal symmetry.

The modular bootstrap relation characterizing this brane is written as

$${}_B\langle B;O|e^{-\pi T H^{(c)}}|B;L\rangle_{AB} = \chi_{L\,(-+)}(it)\,\widehat{\chi_{\mathbf{G}}}(it,0)\,, \tag{4.58}$$

which fixes the overall normalizations in (4.57). Here $\chi_{L\,(-+)}(\tau)$ is the twisted minimal character in M_{N-2}-sector defined in (D.5), and we have introduced a function

$$\widehat{\chi_{\mathbf{G}}}(\tau,z) \equiv \sum_{r\in\mathbf{Z}_N} \chi_{\mathbf{G}}(2r;\tau,z) = q^{-\frac{1}{4N}} \sum_{n\in\mathbf{Z}} \frac{(1-q)q^{\frac{n^2}{N}+n-\frac{1}{2}} e^{2\pi i z\left(\frac{2}{N}n+1\right)}}{\left(1+e^{2\pi i z}q^{n+\frac{1}{2}}\right)\left(1+e^{2\pi i z}q^{n-\frac{1}{2}}\right)} \frac{\theta_3(\tau,z)}{\eta(\tau)^3}\,, \tag{4.59}$$

which is the sum of irreducible graviton character over the whole spectral flows.

Note that (4.57) is naturally regarded as the $\mathbb{Z}_N$-extension of the non-BPS branes in the flat background [54];

$$|Dp\rangle_{\text{non-BPS}} = \frac{1}{\sqrt{2}}\left(|Dp\rangle' + \overline{|Dp\rangle}'\right), \tag{4.60}$$

where $|Dp\rangle'$ ($\overline{|Dp\rangle}'$) expresses the boundary state for the 'wrong-dimensional' BPS (anti-)Dp-brane. Since each term of R.H.S is not compatible with the GSO condition, the boundary state $|Dp\rangle_{\text{non-BPS}}$ cannot be decomposed into constituent branes. In the same sense our non-BPS brane (4.57) is irreducible and not decomposable to constituent boundary states.

It is straightforward to work out cylinder amplitudes and one can show that Cardy condition is always satisfied for the non-BPS brane (4.57). Namely, all the overlaps are interpreted as correct open string one-loop amplitudes. The overlaps with compact BPS branes are easy to evaluate in a similar manner as (4.58), and we also find

$${}_{AB}\langle B;L_1|e^{-\pi T H^{(c)}}|B;L_2\rangle_{AB} = \sum_L N^L_{L_1,L_2}\,\widehat{\mathrm{ch}_L}(it,0)\,\widehat{\chi_{\mathbf{G}}}(it,0)\,, \tag{4.61}$$

where $\widehat{\chi_{\mathbf{G}}}(it,0)$ is defined in (4.59), and also we set

$$\widehat{\mathrm{ch}_\ell}(\tau,z) \equiv \sum_{r\in\mathbf{Z}_N} \mathrm{ch}^{(\mathrm{NS})}_{\ell,\ell+2r}(\tau,z) = \sum_{s\in\mathbf{Z}_{N-2}} c^{(N-2)}_{\ell,\ell+2s}(\tau)\,\Theta_{2\ell+2Ns,N-2}\left(\frac{\tau}{2N},\frac{z}{N}\right)\,. \tag{4.62}$$

Here $c^{(k)}_{\ell,m}(\tau)$ denotes the level k string function of SU(2). Note that the open string channel includes states with fractional U(1)-charges even in the self-overlap cases, suggesting that this boundary state really describes non-BPS D-branes.

To clarify the non-BPS nature it is important to examine the RR-sectors of boundary states. It is easy to construct the RR-counterpart of (4.57). However, we have to take account of the compatibility with the charge integrality and GSO projection together with the Minkowski part $\mathbb{R}^{5,1}$. Let us now recall the formula for U(1)-charges in the R-sector (4.21) ,

$$M_{N-2}\text{-sector}:\ \frac{1}{2}+\frac{m'}{N}, \qquad L_{N,1}\text{-sector}:\ \frac{1}{2}+\frac{n'}{N}\,. \tag{4.63}$$

As we noted above, the sum over r in (4.57) forces the fractional parts of the U(1)-charges $m'/N, n'/N$ to vanish and hence U(1)-charge becomes 1/2 in each R-sector.

When the boundary condition is flipped from B to A-type in the M_{N-2}-sector, U(1)-charge of the right mover changes from 1/2 to −1/2 and there is a net change of U(1)-charge by 1. Then a compensating change must happen in the flat sector along $\mathbb{R}^{5,1}$. One has to shift the dimension of the brane by one and obtains a brane with a wrong dimension $|Dp'\rangle'$ (p' is even(odd) for type IIA (IIB) string theory). Note that in the NS5-brane background a BPS brane has odd dimensions extended in the direction transverse to NS5-brane and thus has odd (even) dimensions extended along the NS5-branes in type IIA (IIB) theory. Non-BPS brane is then given by

$$|Dp'\rangle'^{(\mathrm{NS})} \otimes |B;L\rangle_{AB}^{(\mathrm{NS})} \pm |Dp'\rangle'^{(\mathrm{R})} \otimes |B;L\rangle_{AB}^{(\mathrm{R})} . \tag{4.64}$$

In the present example it is also possible to construct a non-BPS brane of the type known in the flat space-time

$$\sqrt{2}|Dp\rangle^{(\mathrm{NS})} \otimes |B;L\rangle_{AB}^{(\mathrm{NS})} . \tag{4.65}$$

Here p is odd (even) in the type IIA (IIB) theory. The second one (4.65) describes an overall wrong-dimensional brane without RR-component and the overall factor $\sqrt{2}$ is necessary by the same reason as in the flat case. The first one (4.64) is more interesting and characteristic for this conformal system. It is an overall *correct* dimensional brane and has a non-vanishing RR-component. However, the boundary wave functions for the RR-ground states always vanish, implying they have no RR-charges (vanishing periods in other words). In fact, as is obvious from the construction, they have the vanishing $\mathbb{Z}_N$-brane charge valued in the twisted K-group (see e.g. [55]). This type of branes also break the space-time SUSY completely, and one can show that their self-overlaps always include open string tachyons. In fact they always contain a contribution of M_{N-2}-sector

$$\frac{1}{2}\left(\mathrm{ch}_{0,2}^{(\mathrm{NS})}(it,0) - \mathrm{ch}_{0,2}^{(\widetilde{\mathrm{NS}})}(it,0)\right) \equiv \frac{1}{2}\left(\mathrm{ch}_{N-2,N-2}^{(\mathrm{NS})}(it,0) + \mathrm{ch}_{N-2,N-2}^{(\widetilde{\mathrm{NS}})}(it,0)\right), \tag{4.66}$$

in the open string amplitudes, where the second term originates from the RR-boundary states. In the second equality we used the formula (B.6). It yields the leading IR behavior; $\sim e^{-2\pi t(h-1/2)}$ with

$$h = \frac{1}{2} - \frac{1}{N} . \tag{4.67}$$

We have thus found open string tachyon modes for any value $N \geq 2$. We also remark that the involution $L \to N-2-L$ flips the sign in front of the RR-component in (4.64) as in the BPS branes. Hence it is actually enough to only consider the plus sign in (4.64).

We finally make a few comments:

1. In the recent paper [32] D-brane configurations in NS5-backgrounds breaking/not breaking the space-time SUSY have been investigated in detail by means of the DBI action and an interesting geometrical interpretation of tachyon condensation in the non-BPS branes has been proposed in the Little String Theory (LST) [14]. Among other things, it is claimed that the BPS branes lying along the NS5 and breaking the space-time SUSY completely, should be interpreted as the non-BPS branes in LST. It will be interesting to compare this type of branes with our boundary states for the

compact non-BPS branes of the first type (4.64). It is shown in [32] that the open string modes describing the positions of these D-branes transverse to NS5 become tachyonic and have the mass squared[11]

$$\alpha' M_T^2 = -\frac{1}{N} \,. \tag{4.68}$$

This coincides precisely with our calculation of tachyon mass (4.67).

2. We can also consider extensions to more general models and also the case of non-compact branes. The consistency with the GSO projection in RR-sector is the only non-trivial point. For instance, let us consider the models with $N_M = 1$, $N_L \geq 1$, which are identified with the ALE fibrations as we addressed before, and focus on the compact non-BPS branes. We then find

   - If $\gamma (\equiv \hat{c} - N_M - N_L)$ is even, we have a similar situation as in the NS5 case above. Namely, we have two types of non-BPS branes with/without the RR-components such as (4.64), (4.65).
   - If γ is odd, the boundary states of the type (4.64) is not allowed since the charge integrality condition in the R-sector (4.22) is not satisfied (recall that L.H.S of (4.22) vanishes due to r-summation) and R-sector does not appear in the boundary state. We have only the non-BPS branes without RR-components as in the flat background.

5. Conclusions

In this paper we have discussed various aspects of string theory compactification on non-compact Calabi-Yau manifolds by making use of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ supercoset theories coupled to $\mathcal{N}=2$ minimal models. We have used the extended characters of $\mathcal{N}=2$ SCA for the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ theory together with the irreducible characters for the minimal model and determined the closed string massless spectrum, open string Witten index and (non-) BPS boundary states.

Important aspect of the massless spectrum is the following: at our non-compact Gepner points Calabi-Yau 3 and 4-folds possess only (a,c) or (c,a)-type massless states, while (c,c) or (a,a)-type states are absent in the spectrum. Thus the theory possesses only Kähler structure deformations. In the T-dual $\mathcal{N}=2$ Liouville description the theory possesses only complex structure deformations corresponding to the special lagrangian cycles.

This is the characteristic feature of the space-time with a conifold singularity and thus our models describe generalized conifold singularities in CY 3 and 4-folds: $\mathcal{N}=2$ Liouville theories describe their deformations while $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ theories describe their resolutions.

[11] Although the S^1-compactification is taken in [32], the tachyon mass given there does not depend on the compactification radius.

On the other hand, in the case of K3 surface our models possesses equal numbers of (a,c), (c,a), (a,a) and (c,c) states, which corresponds to the characteristic feature of ADE type (hyperKähler) singularities.

We have also studied the spectrum of D-branes in our non-compact models: only the B-type branes are allowed as compact branes (or only the A-branes are possible in the $\mathcal{N}=2$ Liouville theory) and the open string Hilbert space describing compact branes consists of extended graviton representations of $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$-sector and representations of the minimal sector. We have compared the spectra of compact branes with those of the massless states in the closed string sector. Some of the BPS branes (homology cycles) are not associated with massless states: corresponding space-time fields are frozen due to non-normalizability of the wave function. We have also seen that the cylinder amplitudes of $\widetilde{\mathrm{R}}$-sector reproduce expected intersection numbers of vanishing cycles.

The Cardy states describing non-BPS D-branes are also discussed. They are expressed by boundary states breaking the $\mathcal{N}=2$ superconformal symmetry and identified as extensions of the "unstable B-branes" in the SU(2)-WZW model. They may possess (massive) RR-components contrary to non-BPS branes in the flat background.

Geometry of the deformed side of the conifold is relatively easy to study due to the absence of quantum corrections and one can study the lagrangian cycles using Liouville theory. On the other hand, possible quantum corrections make the geometry of resolved conifold difficult to understand. Even the dimensionality of the branes may not be well-defined. It is interesting to see if the description by means of the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ theory will help our understanding of the geometry of resolved conifold.

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A. Notations

We here summarize our notations of theta functions. We set $q \equiv e^{2\pi i\tau}$ and $y \equiv e^{2\pi i z}$,

$$
\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \\
&\equiv 2\sin(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m), \\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\cos(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m)\,, \\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}y^n \equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2})\,,
\end{aligned}
$$

$$\theta_4(\tau,z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2}), \qquad \text{(A.1)}$$

$$\Theta_{m,k}(\tau,z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2} y^{k(n+\frac{m}{2k})}. \qquad \text{(A.2)}$$

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n). \qquad \text{(A.3)}$$

B. Character formulas of $\mathcal{N}=2$ minimal model

The character formulas of the level k $\mathcal{N}=2$ minimal model ($\hat{c}=k/(k+2)$) are described as the branching functions of the Kazama-Suzuki coset $\dfrac{\mathrm{SU}(2)_k \times \mathrm{U}(1)_2}{\mathrm{U}(1)_{k+2}}$ defined by

$$\chi_\ell^{(k)}(\tau,w)\Theta_{s,2}(\tau,w-z) = \sum_{\substack{m\in\mathbf{Z}_{2(k+2)} \\ \ell+m+s\in 2\mathbf{Z}}} \chi_m^{\ell,s}(\tau,z)\Theta_{m,k+2}\left(\tau, w-\frac{2z}{(k+2)}\right),$$

$$\chi_m^{\ell,s}(\tau,z) \equiv 0\,, \qquad \text{for } \ell+m+s \in 2\mathbb{Z}+1\,, \qquad \text{(B.1)}$$

where $\chi_\ell^{(k)}(\tau,z)$ is the spin $\ell/2$ character of $\mathrm{SU}(2)_k$;

$$\chi_\ell^{(k)}(\tau,z) = \frac{\Theta_{\ell+1,k+2}(\tau,z)-\Theta_{-\ell-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z)-\Theta_{-1,2}(\tau,z)} \equiv \sum_{m\in\mathbf{Z}_{2k}} c_{\ell,m}^{(k)}(\tau)\Theta_{m,k}(\tau,z). \qquad \text{(B.2)}$$

The branching function $\chi_m^{\ell,s}(\tau,z)$ is explicitly calculated as follows (see, e.g. [42]);

$$\chi_m^{\ell,s}(\tau,z) = \sum_{r\in\mathbf{Z}_k} c_{\ell,m-s+4r}^{(k)}(\tau)\Theta_{2m+(k+2)(-s+4r),2k(k+2)}\left(\tau,\frac{z}{(k+2)}\right). \qquad \text{(B.3)}$$

Then, the character formulas of unitary representations are written as

$$\begin{aligned}
\mathrm{ch}_{\ell,m}^{(\mathrm{NS})}(\tau,z) &= \chi_m^{\ell,0}(\tau,z)+\chi_m^{\ell,2}(\tau,z)\,,\\
\mathrm{ch}_{\ell,m}^{(\widetilde{\mathrm{NS}})}(\tau,z) &= \chi_m^{\ell,0}(\tau,z)-\chi_m^{\ell,2}(\tau,z) \equiv e^{-i\pi\frac{m}{k+2}}\mathrm{ch}_{\ell,m}^{(\mathrm{NS})}\left(\tau,z+\frac{1}{2}\right),\\
\mathrm{ch}_{\ell,m}^{(\mathrm{R})}(\tau,z) &= \chi_m^{\ell,1}(\tau,z)+\chi_m^{\ell,3}(\tau,z) \equiv q^{\frac{k}{8(k+2)}}y^{\frac{k}{2(k+2)}}\mathrm{ch}_{\ell,m+1}^{(\mathrm{NS})}\left(\tau,z+\frac{\tau}{2}\right),\\
\mathrm{ch}_{\ell,m}^{(\widetilde{\mathrm{R}})}(\tau,z) &= \chi_m^{\ell,1}(\tau,z)-\chi_m^{\ell,3}(\tau,z) \equiv -e^{-i\pi\frac{m+1}{k+2}}q^{\frac{k}{8(k+2)}}y^{\frac{k}{2(k+2)}}\mathrm{ch}_{\ell,m+1}^{(\mathrm{NS})}\left(\tau,z+\frac{1}{2}+\frac{\tau}{2}\right). \qquad \text{(B.4)}
\end{aligned}$$

By definition, we may restrict to $\ell+m\in 2\mathbb{Z}$ for the NS and $\widetilde{\mathrm{NS}}$ sectors, and to $\ell+m\in 2\mathbb{Z}+1$ for the R and $\widetilde{\mathrm{R}}$ sectors. It is convenient to define $\mathrm{ch}_*^{(\sigma)}(\tau,z)\equiv 0$ unless these conditions for ℓ, m are satisfied. Note that the character identity ("field identification") holds

$$\chi_{m+k+2}^{k-\ell,s+2}(\tau,z) = \chi_m^{\ell,s}(\tau,z)\,, \qquad \text{(B.5)}$$

or equivalently,

$$\begin{aligned}
\mathrm{ch}_{k-\ell,m+k+2}^{(\sigma)}(\tau,z) &= \mathrm{ch}_{\ell,m}^{(\sigma)}(\tau,z)\,, \qquad (\sigma=\mathrm{NS},\,\mathrm{R})\,,\\
\mathrm{ch}_{k-\ell,m+k+2}^{(\sigma)}(\tau,z) &= -\mathrm{ch}_{\ell,m}^{(\sigma)}(\tau,z)\,, \qquad (\sigma=\widetilde{\mathrm{NS}},\,\widetilde{\mathrm{R}})\,. \qquad \text{(B.6)}
\end{aligned}$$

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We next present the modular transformation formulas. To this aim it is convenient to introduce the notations

$$T \cdot \mathrm{NS} = \widetilde{\mathrm{NS}}\,, \qquad T \cdot \widetilde{\mathrm{NS}} = \mathrm{NS}\,, \qquad T \cdot \mathrm{R} = \mathrm{R}\,, \qquad T \cdot \widetilde{\mathrm{R}} = \widetilde{\mathrm{R}}\,, \tag{B.7}$$

$$S \cdot \mathrm{NS} = \mathrm{NS}\,, \qquad S \cdot \widetilde{\mathrm{NS}} = \mathrm{R}\,, \qquad S \cdot \mathrm{R} = \widetilde{\mathrm{NS}}\,, \qquad S \cdot \widetilde{\mathrm{R}} = \widetilde{\mathrm{R}}\,. \tag{B.8}$$

We also define

$$\kappa(\sigma) = \begin{cases} 1 & \sigma = \mathrm{NS},\, \widetilde{\mathrm{NS}},\, \mathrm{R} \\ -i & \sigma = \widetilde{\mathrm{R}} \end{cases}, \qquad \nu(\sigma) = \begin{cases} 0 & \sigma = \mathrm{NS},\, \widetilde{\mathrm{NS}} \\ 1 & \sigma = \mathrm{R},\, \widetilde{\mathrm{R}} \end{cases}. \tag{B.9}$$

Modular transformations are given by

$$\chi_m^{\ell,s}(\tau+1,z) = e^{2\pi i\left(h^{(\mathrm{NS})}(\ell,m)+\frac{s^2}{8}-\frac{k}{8(k+2)}\right)}\,\chi_m^{\ell,s}(\tau,z)\,, \tag{B.10}$$

$$\chi_m^{\ell,s}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{i\pi\frac{k}{k+2}\frac{z^2}{\tau}}\sum_{\ell'=0}^{k}\sum_{m\in\mathbf{Z}_{2(k+2)}}\sum_{s\in\mathbf{Z}_4} S_{\ell,m}^{\ell',m'}\,\frac{1}{2}e^{-i\pi\frac{ss'}{2}}\,\chi_{m'}^{\ell',s'}(\tau,z)\,, \tag{B.11}$$

where $h^{(\mathrm{NS})}(\ell,m) \equiv \dfrac{\ell(\ell+2)-m^2}{4(k+2)}$ and $S_{\ell,m}^{\ell',m'}$ is the modular coefficients

$$S_{\ell,m}^{\ell',m'} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(\ell+1)(\ell'+1)}{k+2}\right)\cdot\frac{1}{\sqrt{2(k+2)}}e^{2\pi i\frac{mm'}{2(k+2)}}\,. \tag{B.12}$$

Equivalently,

$$\mathrm{ch}_{\ell,m}^{(\sigma)}(\tau+1,z) = e^{2\pi i\left(h^{(\sigma)}-\frac{k}{8(k+2)}\right)}\,\mathrm{ch}_{\ell,m}^{(T\cdot\sigma)}(\tau,z)\,, \tag{B.13}$$

$$\mathrm{ch}_{\ell,m}^{(\sigma)}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \kappa(\sigma)e^{i\pi\frac{k}{k+2}\frac{z^2}{\tau}}\sum_{\ell'=0}^{k}\sum_{m\in\mathbf{Z}_{2(k+2)}} S_{\ell,m}^{\ell',m'}\,\mathrm{ch}_{\ell',m'}^{(S\cdot\sigma)}(\tau,z)\,, \tag{B.14}$$

where $h^{(\widetilde{\mathrm{NS}})}(\ell,m) = h^{(\mathrm{NS})}(\ell,m)$, $h^{(\mathrm{R})}(\ell,m) = h^{(\widetilde{\mathrm{R}})}(\ell,m) = h^{(\mathrm{NS})}(\ell,m)+1/8$. We note the spectral flow relations ($^{\forall}a, {}^{\forall}b \in \mathbb{Z}$);

$$q^{\frac{k}{2(k+2)}a^2}e^{2\pi i\frac{k}{k+2}az}\,\chi_m^{\ell,s}(\tau,z+a\tau+b) = e^{-i\pi sb}e^{2\pi i\frac{m}{k+2}b}\,\chi_{m-2a}^{\ell,s-2a}(\tau,z)\,. \tag{B.15}$$

Especially, they have the (quasi-)periodicity of period $k+2$ with respect to z. We also note the formula of Witten index

$$\lim_{z\to 0}\mathrm{ch}_{\ell,m}^{(\widetilde{\mathrm{R}})}(\tau,z) = \delta_{m,\ell+1}^{(2(k+2))} - \delta_{m,-(\ell+1)}^{(2(k+2))}\,. \tag{B.16}$$

C. Extended characters and their modular properties

In this appendix we summarize useful properties of the extended characters (2.12), (2.14) and (2.17) in $L_{N,K}$-sector, i.e. the $\mathrm{SL}(2;\mathbb{R})/\mathrm{U}(1)$ supercoset with $k = N/K$ ($N, K \in \mathbb{Z}_{>0}$)

introduced in [1, 4]. They are explicitly written as

$$\chi_{\mathbf{c}}{}^{(\mathrm{NS})}(p,m;\tau,z) = q^{\frac{p^2}{2}}\Theta_{m,NK}\left(\tau,\frac{2z}{N}\right)\frac{\theta_3(\tau,z)}{\eta(\tau)^3}, \qquad (C.1)$$

$$\chi_{\mathbf{d}}{}^{(\mathrm{NS})}(s,s+2Kr;\tau,z) = \sum_{n\in\mathbf{Z}}\frac{\left(yq^{N\left(n+\frac{2r+1}{2N}\right)}\right)^{\frac{s-K}{N}} y^{2K\left(n+\frac{2r+1}{2N}\right)}q^{NK\left(n+\frac{2r+1}{2N}\right)^2}}{1+yq^{N\left(n+\frac{2r+1}{2N}\right)}}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}, \qquad (C.2)$$

$$\chi_{\mathbf{d}}{}^{(\mathrm{NS})}(s,m;\tau,z) \equiv 0, \qquad (m-s\notin 2K\mathbb{Z}),$$

$$\chi_0^{(\mathrm{NS})}(2Kr;\tau,z) = q^{-\frac{K}{4N}}\sum_{n\in\mathbf{Z}}\frac{(1-q)q^{NK\left(n+\frac{r}{N}\right)^2+N\left(n+\frac{2r-1}{2N}\right)}y^{2K\left(n+\frac{r}{N}\right)+1}}{\left(1+yq^{N\left(n+\frac{2r+1}{2N}\right)}\right)\left(1+yq^{N\left(n+\frac{2r-1}{2N}\right)}\right)}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}$$

$$\chi_0^{(\mathrm{NS})}(m;\tau,z) \equiv 0, \qquad (m\notin 2K\mathbb{Z}). \qquad (C.3)$$

Expressions for other spin structures are readily derived from the definitions (2.19).

The formulas of Witten indices are given as

$$\chi_{\mathbf{c}}{}^{(\widetilde{\mathrm{R}})}(p,m;\tau,0)=0, \quad \chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s,m;\tau,0)=-\delta^{(2NK)}_{m,s-K}, \quad \chi_0^{(\widetilde{\mathrm{R}})}(m;\tau,0)=\delta^{(2NK)}_{m,K}-\delta^{(2NK)}_{m,-K}. \qquad (C.4)$$

The following relation of spectral flow is obvious from the definitions ($^\forall a,b\in\mathbb{Z}$);

$$q^{\frac{\hat{c}}{2}a^2}e^{2\pi i\hat{c}az}\chi_*^{(\sigma)}(*,m;\tau,z+a\tau+b)=\epsilon_\sigma(a,b)e^{2\pi i\frac{m}{N}b}\chi_*^{(\sigma)}(*,m+2Ka;\tau,z), \qquad (C.5)$$

where the sign factor $\epsilon_\sigma(a,b)$ is given by $\epsilon_{\mathrm{NS}}(a,b)=1$, $\epsilon_{\widetilde{\mathrm{NS}}}(a,b)=(-1)^a$, $\epsilon_{\mathrm{R}}(a,b)=(-1)^b$, $\epsilon_{\widetilde{\mathrm{R}}}(a,b)=(-1)^{a+b}$, respectively. Especially, the extended characters have the (quasi-)periodicity with period N under the spectral flows. We also note the "charge conjugation relations" of extended characters;

$$\chi_{\mathbf{c}}{}^{(\sigma)}(p,m;\tau,-z) = \chi_{\mathbf{c}}{}^{(\sigma)}(p,-m;\tau,z), \qquad (\sigma=\mathrm{NS},\ \widetilde{\mathrm{NS}},\ \mathrm{R}),$$

$$\chi_{\mathbf{c}}{}^{(\widetilde{\mathrm{R}})}(p,m;\tau,-z) = -\chi_{\mathbf{c}}{}^{(\widetilde{\mathrm{R}})}(p,-m;\tau,z), \qquad (C.6)$$

$$\chi_{\mathbf{d}}{}^{(\sigma)}(s,m;\tau,-z) = \chi_{\mathbf{d}}{}^{(\sigma)}(N+2K-s,N-m;\tau,z), \qquad (\sigma=\mathrm{NS},\ \mathrm{R},\ \widetilde{\mathrm{R}}),$$

$$\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{NS}})}(s,m;\tau,-z) = -\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{NS}})}(N+2K-s,N-m;\tau,z), \qquad (C.7)$$

$$\chi_0^{(\sigma)}(m;\tau,-z) = \chi_0^{(\sigma)}(-m;\tau,z), \qquad (\sigma=\mathrm{NS},\ \widetilde{\mathrm{NS}},\ \mathrm{R}),$$

$$\chi_0^{(\widetilde{\mathrm{R}})}(m;\tau,-z) = -\chi_0^{(\widetilde{\mathrm{R}})}(-m;\tau,z). \qquad (C.8)$$

The following identities are also useful ($r\in\mathbb{Z}_N$);

$$\chi_0^{(\sigma)}(K(2r+\nu(\sigma));\tau,z)\pm$$
$$\pm\chi_{\mathbf{d}}{}^{(\sigma)}(N,N+K(2r+\nu(\sigma));\tau,z)+\chi_{\mathbf{d}}{}^{(\sigma)}(2K,K(2r+\nu(\sigma));\tau,z)=$$
$$=\chi_{\mathbf{c}}{}^{(\sigma)}\left(i\sqrt{\frac{K}{2N}},K(2r+\nu(\sigma));\tau,z\right), \qquad (C.9)$$

$$\chi_{\mathbf{d}}{}^{(\sigma)}(K,m;\tau,z)\pm\chi_{\mathbf{d}}{}^{(\sigma)}(N+K,m+N;\tau,z)=\chi_{\mathbf{c}}{}^{(\sigma)}(p=0,m;\tau,z), \quad (\sigma=\mathrm{NS},\ \mathrm{R}), \qquad (C.10)$$

where one chooses $+$ $(-)$ sign for spin structures NS and R ($\widetilde{\mathrm{NS}},\widetilde{\mathrm{R}}$).

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Let us next present the modular transformation formulas of the extended characters. The T-transformation formulas are quite easy;

$$\chi_*^{(\sigma)}(*,m;\tau+1,z) = e^{2\pi i (h^{(\sigma)}(*,m)-\hat{c}/8)}\,\chi_*^{(T\cdot\sigma)}(*,m;\tau,z)\,, \tag{C.11}$$

where the conformal weight $h^{(\sigma)}(*,m)$ can be read off from (2.13), (2.16), (2.18) and also the formula (2.20). The S-transformation formulas are given in [1], and written as

$$\begin{aligned}
&{\chi_{\mathbf{c}}}^{(\sigma)}\left(p,m;-\frac{1}{\tau},\frac{z}{\tau}\right) = \kappa(\sigma)e^{i\pi\hat{c}\frac{z^2}{\tau}}\sqrt{\frac{2}{NK}}\times\\
&\qquad\times\sum_{m'\in\mathbf{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}}\int_0^\infty dp'\,\cos(2\pi pp')\,{\chi_{\mathbf{c}}}^{(S\cdot\sigma)}(p',m';\tau,z)\,,
\end{aligned}\tag{C.12}$$

$$\begin{aligned}
&{\chi_{\mathbf{d}}}^{(\sigma)}\left(s,m;-\frac{1}{\tau},\frac{z}{\tau}\right) =\\
&= \kappa(\sigma)e^{i\pi\hat{c}\frac{z^2}{\tau}}\left[\frac{1}{\sqrt{2NK}}\sum_{m'\in\mathbf{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}}\times\right.\\
&\quad\times\int_0^\infty dp'\,\frac{\cosh\left(2\pi\frac{N-(s-K)}{\sqrt{2NK}}p'\right)+e^{2\pi i\left(\frac{m'}{2K}-\frac{1}{2}\nu(S\cdot\sigma)\right)}\cosh\left(2\pi\frac{s-K}{\sqrt{2NK}}p'\right)}{2\left|\cosh\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}-\frac{i}{2}\nu(S\cdot\sigma)\right)\right|^2}\times\\
&\quad\times{\chi_{\mathbf{c}}}^{(S\cdot\sigma)}(p',m';\tau,z)+\\
&\quad+\frac{i}{N}\sum_{s'=K+1}^{N+K-1}\sum_{m'\in\mathbf{Z}_{2NK}} e^{2\pi i\frac{(s-K)(s'-K)-mm'}{2NK}}\,{\chi_{\mathbf{d}}}^{(S\cdot\sigma)}(s',m';\tau,z)+\\
&\quad+\frac{i}{2N}\sum_{m'\in\mathbf{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}}\left\{{\chi_{\mathbf{d}}}^{(S\cdot\sigma)}(K,m';\tau,z)\right.\\
&\qquad\left.\left.-\,e^{i\pi\nu(\sigma)}\,{\chi_{\mathbf{d}}}^{(S\cdot\sigma)}(N+K,N+m';\tau,z)\right\}\right],
\end{aligned}\tag{C.13}$$

$$\begin{aligned}
&\chi_0^{(\sigma)}\left(m;-\frac{1}{\tau},\frac{z}{\tau}\right) =\\
&= \kappa(\sigma)e^{i\pi\hat{c}\frac{z^2}{\tau}}\left[\frac{1}{\sqrt{2NK}}\sum_{m'\in\mathbf{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}}\times\right.\\
&\quad\times\int_0^\infty dp'\,\frac{\sinh\left(\pi\sqrt{\frac{2K}{N}}p'\right)\sinh\left(\pi\sqrt{\frac{2N}{K}}p'\right)}{\left|\cosh\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}-\frac{i}{2}\nu(S\cdot\sigma)\right)\right|^2}\,{\chi_{\mathbf{c}}}^{(S\cdot\sigma)}(p',m';\tau,z)+\\
&\quad\left.+\frac{2}{N}\sum_{s'=K+1}^{N+K-1}\sum_{m'\in\mathbf{Z}_{2NK}}\sin\left(\frac{\pi(s'-K)}{N}\right)e^{-2\pi i\frac{mm'}{2NK}}\,{\chi_{\mathbf{d}}}^{(S\cdot\sigma)}(s',m';\tau,z)\right].
\end{aligned}\tag{C.14}$$

In these expressions $T\cdot\sigma$, $S\cdot\sigma$, $\kappa(\sigma)$ and $\nu(\sigma)$ are defined in (B.7), (B.8), (B.9). Note that these modular transformation formulas, especially the discrete terms appearing in them, are really consistent with the formulas of Witten index (C.4).

It is often useful to extend the S-transformation formula for the massive character (C.12) to the case of continuous U(1)-charge;

$$\chi_{\mathbf{c}}{}^{(\sigma)}\left(p,\omega;-\frac{1}{\tau},\frac{z}{\tau}\right)=\kappa(\sigma)e^{i\pi\hat{c}\frac{z^2}{\tau}}\sqrt{\frac{2}{NK}}\sum_{m'\in\mathbf{Z}_{2NK}}\int_0^\infty dp'\cos\left(2\pi pp'\right)\chi_{\mathbf{c}}{}^{(S\cdot\sigma)}\left(p',m';\tau,z,\omega\right). \tag{C.15}$$

Here the l.h.s. is defined by formally replacing the discrete parameter $m \in \mathbb{Z}_{2NK}$ with a continuous one $0 \leq \omega < 2NK$ in (2.12). In the r.h.s. we introduced

$$\chi_{\mathbf{c}}{}^{(\sigma)}(p, m; \tau, z, w) = q^{\frac{p^2}{2}} \Theta_{m,NK}\left(\tau, \frac{2z}{N} - \frac{w}{NK}\right) \frac{\theta_{[\sigma]}(\tau, z)}{\eta(\tau)^3}\,, \tag{C.16}$$

where we set $\theta_{[\sigma]} = \theta_3, \theta_4, \theta_2, i\theta_1$ for $\sigma = \mathrm{NS}, \widetilde{\mathrm{NS}}, \mathrm{R}, \widetilde{\mathrm{R}}$. $\chi_{\mathbf{c}}{}^{(\sigma)}(p, m; \tau, z, w)$ is the extension of extended massive character with one parameter $0 \leq w < 2NK$ that characterizes the relative phases of irreducible characters in the sum over spectral flow. Note that the R.H.S in (C.15) contains only contributions with discrete U(1)-charges.

D. Twisted $\mathcal{N} = 2$ characters

The twisted $\mathcal{N} = 2$ characters are defined by the twisting with respect to a $\mathbb{Z}_2$-automorphism in the $\mathcal{N} = 2$ SCA;

$$\sigma \;:\; T \longrightarrow T, \qquad J \longrightarrow -J, \qquad G^{\pm} \longrightarrow G^{\mp}\,. \tag{D.1}$$

We denote the twisted characters as $\mathrm{ch}^{(*)}_{(S,T)}$ where $*$ expresses the spin structure and $S, T = \pm$ express the spatial and temporal boundary conditions of the σ-twist. The relevant quantum number is only the conformal weight, since the σ-twisting leaves only the states with vanishing U(1)-charge. It is easy to see the following identities (see e.g. [56]);

$$\mathrm{ch}^{(\mathrm{NS})}_{(+,-)}(\tau) = \mathrm{ch}^{(\widetilde{\mathrm{NS}})}_{(+,-)}(\tau)\,, \quad \mathrm{ch}^{(\mathrm{NS})}_{(-,+)}(\tau) = \mathrm{ch}^{(\mathrm{R})}_{(-,+)}(\tau)\,, \quad \mathrm{ch}^{(\widetilde{\mathrm{NS}})}_{(-,-)}(\tau) = \mathrm{ch}^{(\mathrm{R})}_{(-,-)}(\tau)\,, \tag{D.2}$$

$$\mathrm{ch}^{(\mathrm{R})}_{(+,-)}(\tau) = \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{(+,-)}(\tau)\,, \quad \mathrm{ch}^{(\widetilde{\mathrm{NS}})}_{(-,+)}(\tau) = \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{(-,+)}(\tau)\,, \quad \mathrm{ch}^{(\mathrm{NS})}_{(-,-)}(\tau) = \mathrm{ch}^{(\widetilde{\mathrm{R}})}_{(-,-)}(\tau)\,. \tag{D.3}$$

All the characters in the second line (D.3) actually vanish due to fermion zero modes, and we are left only three non-trivial twisted characters (D.2), which we denote as $\chi_{(+,-)}(\tau)$, $\chi_{(-,+)}(\tau)$ and $\chi_{(-,-)}(\tau)$.

The twisted massive characters in any $\mathcal{N} = 2$ SCFT's with $\hat{c} > 1$ are essentially trivial. We can readily calculate them as

$$\begin{aligned}
\chi_{(+,-)}(p;\tau) &= \frac{2q^{\frac{p^2}{2}}}{\theta_2(\tau)}\,, \qquad \left(h = \frac{p^2}{2} + \frac{\hat{c}-1}{8}\right), \\
\chi_{(-,+)}(p;\tau) &= \frac{2q^{\frac{p^2}{2}}}{\theta_4(\tau)}\,, \qquad \left(h = \frac{p^2}{2} + \frac{\hat{c}}{8}\right), \\
\chi_{(-,-)}(p;\tau) &= \frac{2q^{\frac{p^2}{2}}}{\theta_3(\tau)}\,, \qquad \left(h = \frac{p^2}{2} + \frac{\hat{c}}{8}\right).
\end{aligned} \tag{D.4}$$

In the minimal model M_k the twisted characters are much more involved. The relevant formulas are summarized in [56] based on the results [57, 58, 59];

$$\chi_{\ell\,(+,-)}(\tau) = \begin{cases} \dfrac{2}{\theta_2(\tau)}\left(\Theta_{2(\ell+1),4(k+2)}(\tau) + (-1)^k \Theta_{2(\ell+1)+4(k+2),4(k+2)}(\tau)\right) & (\ell \;:\; \text{even}), \\ 0 & (\ell \;:\; \text{odd}). \end{cases}$$

$$\begin{aligned}
\chi_{\ell\,(-,+)}(\tau) &= \frac{1}{\theta_4(\tau)}\left(\Theta_{\ell+1-\frac{k+2}{2},k+2}(\tau)-\Theta_{-(\ell+1)-\frac{k+2}{2},k+2}(\tau)\right)\\
&= \frac{1}{\theta_4(\tau)}\big(\Theta_{2(\ell+1)-(k+2),4(k+2)}(\tau)+\Theta_{2(\ell+1)+3(k+2),4(k+2)}(\tau)-\\
&\quad -\Theta_{-2(\ell+1)-(k+2),4(k+2)}(\tau)-\Theta_{-2(\ell+1)+3(k+2),4(k+2)}(\tau)\big)\ ,\\
\chi_{\ell\,(-,-)}(\tau) &= \frac{1}{\theta_3(\tau)}\Big(\Theta_{2(\ell+1)-(k+2),4(k+2)}(\tau)+(-1)^k\Theta_{2(\ell+1)+3(k+2),4(k+2)}(\tau)\\
&\quad +(-1)^\ell\Theta_{-2(\ell+1)-(k+2),4(k+2)}(\tau)+(-1)^{k+\ell}\Theta_{-2(\ell+1)+3(k+2),4(k+2)}(\tau)\Big)\,. \qquad \text{(D.5)}
\end{aligned}$$

The first character has the vacuum with

$$h=h_\ell\equiv\frac{\ell(\ell+2)}{4(k+2)}\,,\qquad \text{(the same as the primary of } SU(2)_k)\,, \qquad \text{(D.6)}$$

and, the second and third ones have the vacuum with

$$h=h_\ell^t\equiv\frac{k-2+(k-2\ell)^2}{16(k+2)}+\frac{1}{16}\,. \qquad \text{(D.7)}$$

This vacuum of (D.7) is interpreted as given by the product of the twist field in the U(1)-sector and the "C-disorder field" [58] in the $\mathbb{Z}_k$-parafermion theory [60]. Note that we have $\chi_{k-\ell\,(-,+)}=\chi_{\ell\,(-,+)}$, $\chi_{k-\ell\,(-,-)}=\chi_{\ell\,(-,-)}$. We have to actually identify the corresponding primary fields, only leaving $\ell=0,1,\ldots,\left[\frac{k}{2}\right]$ as independent primary fields.

The modular transformation formulas are written as

$$\begin{aligned}
&\chi_{\ell\,(+,-)}(\tau+1)=e^{2\pi i\left(h_\ell-\frac{k}{8(k+2)}\right)}\,\chi_{\ell\,(+,-)}(\tau)\,,\ \chi_{\ell\,(+,-)}\left(-\frac{1}{\tau}\right)=\sum_{\ell'=0}^{k}(-1)^{\ell/2}S_{\ell,\ell'}\,\chi_{\ell'\,(-,+)}(\tau),\\
&\chi_{\ell\,(-,+)}(\tau+1)=e^{2\pi i\left(h^t_\ell-\frac{k}{8(k+2)}\right)}\,\chi_{\ell\,(-,-)}(\tau)\,,\ \chi_{\ell\,(-,+)}\left(-\frac{1}{\tau}\right)=\sum_{\ell'=0}^{k}S_{\ell,\ell'}(-1)^{\ell'/2}\,\chi_{\ell'\,(+,-)}(\tau)\,,\\
&\chi_{\ell\,(-,-)}(\tau+1)=e^{2\pi i\left(h^t_\ell-\frac{k}{8(k+2)}\right)}\,\chi_{\ell\,(-,+)}(\tau)\,,\ \chi_{\ell\,(-,-)}\left(-\frac{1}{\tau}\right)=(-i)\sum_{\ell'=0}^{k}\widehat{S}_{\ell,\ell'}\,\chi_{\ell'\,(-,-)}(\tau)\,.
\end{aligned} \qquad \text{(D.8)}$$

Here $S_{\ell,\ell'}\equiv\sqrt{\frac{2}{k+2}}\sin\left(\frac{(\ell+1)(\ell'+1)}{k+2}\right)$ is the modular S-matrix of the SU(2) WZW model at level k, and we set $\widehat{S}_{\ell,\ell'}=e^{\frac{\pi i}{2}\left(\ell+\ell'+2-\frac{k+2}{2}\right)}\,S_{\ell,\ell'}$.

E. Extended characters and appell function

In [1] the modular transformation formulas of the massless extended characters in the $L_{N,K}$-sector have been derived using the integration formula presented in [61]. On the other hand, the massless extended characters are closely related with the higher level Appell functions [33, 34], and their modular properties are studied in [34]. In this comparably long appendix we try to rederive the modular transformation formulas of massless extended characters based on the results given in [34]. We shall use the notations $e^{2\pi i\tau}\equiv q$, $e^{-2\pi i\frac{1}{\tau}}\equiv\tilde{q}$ and $e^{2\pi iz}\equiv y$. For our purpose it is the most convenient to concentrate on the $\widetilde{\mathrm{R}}$-sector.

E.1 Preliminaries

The relevant modular transformation formula (C.13) can be rewritten in a more convenient form (for the $\widetilde{\rm R}$-sector);

$$\chi_{\mathbf{d}}{}^{(\widetilde{\rm R})}\left(s,m;-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{i\pi\hat{c}\frac{z^2}{\tau}}\left[-\frac{i}{\sqrt{2NK}}\sum_{m'\in\mathbf{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}}\times\right.$$
$$\times\int_{\mathbf{R}+i0} dp'\,\frac{e^{-2\pi\frac{s-K}{\sqrt{2NK}}p'}}{1-e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)}}\,\chi_{\mathbf{c}}{}^{(\widetilde{\rm R})}(p',m';\tau,z)+$$
$$\left.+\frac{1}{N}\sum_{s'=K}^{N+K-1}\sum_{m'\in\mathbf{Z}_{2NK}} e^{2\pi i\frac{(s-K)(s'-K)-mm'}{2NK}}\,\chi_{\mathbf{d}}{}^{(\widetilde{\rm R})}(s',m';\tau,z)\right], \tag{E.1}$$

The extended massive character $\chi_{\mathbf{c}}{}^{(\widetilde{\rm R})}(p,m)$ is explicitly written as

$$\chi_{\mathbf{c}}{}^{(\widetilde{\rm R})}(p,m;\tau,z)\equiv q^{\frac{p^2}{2}}\Theta_{m,NK}\left(\tau,\frac{2z}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\,. \tag{E.2}$$

On the other hand, the modular transformation formula of the level ℓ Appell function (3.25) is given by [34];

$$\begin{aligned}\mathcal{K}_\ell\left(-\frac{1}{\tau},\frac{\nu}{\tau},\frac{\mu}{\tau}\right) &= \tau e^{i\pi\ell\frac{\nu^2-\mu^2}{\tau}}\mathcal{K}_\ell(\tau,\nu,\mu)+\tau\sum_{a=0}^{\ell-1} e^{i\pi\frac{\ell}{\tau}(\nu+\frac{a}{\ell}\tau)^2}\Phi(\ell\tau,\ell\mu-a\tau)\theta_3(\ell\tau,\ell\nu+a\tau)\\ &\equiv \tau e^{i\pi\ell\frac{\nu^2-\mu^2}{\tau}}\mathcal{K}_\ell(\tau,\nu,\mu)+\tau e^{i\pi\ell\frac{\nu^2}{\tau}}\sum_{a=0}^{\ell-1}\Phi(\ell\tau,\ell\mu-a\tau)\Theta_{a,\frac{\ell}{2}}(\tau,2\nu)\,,\end{aligned} \tag{E.3}$$

where we set

$$\Phi(\tau,\mu)\equiv-\frac{i}{2\sqrt{-i\tau}}-\frac{1}{2}\int_{-\infty}^{\infty}dx\,e^{-\pi x^2}\frac{\sinh(\pi x\sqrt{-i\tau}(1+2\frac{\mu}{\tau}))}{\sinh(\pi x\sqrt{-i\tau})}\,. \tag{E.4}$$

As we see below, it is enough to only consider the $\mu=0$ case. Using the identity

$$\frac{\tilde{q}^{\frac{x'^2}{2}}}{\sqrt{-i\tau}}=\int_{\mathbf{R}+i\xi}dp'\,e^{-2\pi i p'x'}q^{\frac{p'^2}{2}}\,,\qquad({}^\forall\xi\in\mathbb{R})\,, \tag{E.5}$$

and also the standard contour deformation technique, we can rewrite (E.3) in a more convenient form;

$$\mathcal{K}_\ell\left(-\frac{1}{\tau},\frac{\nu}{\tau},0\right)=\tau e^{i\pi\ell\frac{\nu^2}{\tau}}\left[\mathcal{K}_\ell(\tau,\nu,0)-\frac{i}{\sqrt{\ell}}\sum_{a=0}^{\ell-1}\Theta_{a,\frac{\ell}{2}}(\tau,2\nu)\int_{\mathbf{R}+i0}dp'\,\frac{1}{1-e^{-2\pi\left(\frac{p'}{\sqrt{\ell}}+i\frac{a}{\ell}\right)}}q^{\frac{p'^2}{2}}\right]. \tag{E.6}$$

E.2 Derivation of eq. (E.1) from the STT formula (E.6)

Now, we try to derive the modular transformation formula (E.1) from (E.6). The key idea is to make use of the relations between the massless characters and Appell function (3.26), (3.27).

We first focus on the range $K \leq s \leq N+K-1$, and later discuss other values. Based on (3.27), our first task is to evaluate the modular transformation of $\mathcal{K}_{2NK}\left(\tau, \frac{z+a\tau+b}{N}, 0\right)$ by the formula (E.6) $(a, b \in \mathbb{Z}_N)$;

$$
\begin{aligned}
&\mathcal{K}_{2NK}\left(-\frac{1}{\tau}, \frac{1}{N}\left(\frac{z}{\tau}-\frac{a}{\tau}+b\right), 0\right) = \\
&\equiv \mathcal{K}_{2NK}\left(-\frac{1}{\tau}, \frac{1}{\tau}\left(\frac{z+b\tau-a}{N}\right), 0\right) \\
&= \tau e^{i\pi 2NK \frac{1}{\tau}\left(\frac{z+b\tau-a}{N}\right)^2}\left[\mathcal{K}_{2NK}\left(\tau, \frac{z+b\tau-a}{N}, 0\right) - \right. \\
&\qquad\qquad - \frac{i}{\sqrt{2NK}} \sum_{m' \in \mathbf{Z}_{2NK}} \Theta_{m',NK}\left(\tau, \frac{2(z+b\tau-a)}{N}\right) \times \\
&\qquad\qquad \left. \times \int_{\mathbf{R}+i0} dp' \, \frac{1}{1-e^{-2\pi\left(\frac{p'}{\sqrt{2NK}}+i\frac{m'}{2NK}\right)}} \, q^{\frac{p'^2}{2}}\right].
\end{aligned} \tag{E.7}
$$

Using (3.27), we obtain

$$
\begin{aligned}
&\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{NS}})}\left(s, s-K+2Ka; -\frac{1}{\tau}, \frac{z}{\tau}\right) = \\
&= \tilde{q}^{\frac{K}{N}a^2} e^{2\pi i \frac{2K}{N}\frac{az}{\tau}} \frac{1}{N} \sum_{b \in \mathbf{Z}_N} e^{-2\pi i \frac{(s-K)b}{N}} \mathcal{K}_{2NK}\left(-\frac{1}{\tau}, \frac{1}{N}\left(\frac{z}{\tau}-\frac{a}{\tau}+b\right), 0\right) \frac{i\theta_1(-1/\tau, z/\tau)}{\eta(-1/\tau)^3} \\
&= e^{i\pi\hat{c}\frac{z^2}{\tau}} \frac{1}{N} \sum_{b \in \mathbf{Z}_N} q^{\frac{K}{N}b^2} y^{\frac{2K}{N}b} e^{-2\pi i \frac{s-K+2Ka}{N}b} \times \\
&\quad \times \left[\mathcal{K}_{2NK}\left(\tau, \frac{z+b\tau-a}{N}, 0\right) - \right. \\
&\quad - \frac{i}{\sqrt{2NK}} \sum_{m' \in \mathbf{Z}_{2NK}} \Theta_{m',NK}\left(\tau, \frac{2(z+b\tau-a)}{N}\right) \times \\
&\quad \left. \times \int_{\mathbf{R}+i0} dp' \, \frac{1}{1-e^{-2\pi\left(\frac{p'}{\sqrt{2NK}}+i\frac{m'}{2NK}\right)}} \, q^{\frac{p'^2}{2}}\right] \frac{i\theta_1(\tau, z)}{\eta(\tau)^3}, \\
&\equiv \textbf{(dis. term)} + \textbf{(con. term)}.
\end{aligned} \tag{E.8}
$$

The terms **(dis. term)** and **(con. term)** correspond to the first and second terms, which we will separately evaluate.

1. discrete term:

Recalling the relation (3.26), we obtain

$$
\begin{aligned}
\textbf{(dis. term)} &= e^{i\pi\hat{c}\frac{z^2}{\tau}} \frac{1}{N} \sum_{b \in \mathbf{Z}_N} \sum_{s'=K}^{N+K-1} e^{-\frac{2\pi i}{N}\{(s-K)b+(s'-K)a+2Kab\}} \times \\
&\quad \times \chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s', s'-K+2Kb; \tau, z) \\
&= e^{i\pi\hat{c}\frac{z^2}{\tau}} \frac{1}{N} \sum_{m' \in \mathbf{Z}_{2NK}} \sum_{s'=K}^{N+K-1} e^{2\pi i \frac{(s-K)(s'-K)-mm'}{2NK}} \times \\
&\quad \times \chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s', m'; \tau, z).
\end{aligned} \tag{E.9}
$$

In the second line we set $m = s - K + 2Ka$, $m' = s' - K + 2Kb$.

2. continuous term:

We first note

$$q^{\frac{K}{N}b^2} y^{\frac{2K}{N}b}\, \Theta_{m',NK}\left(\tau, \frac{2(z+b\tau-a)}{N}\right) = e^{-2\pi i \frac{am'}{N}}\, \Theta_{m'+2Kb,NK}\left(\tau, \frac{2z}{N}\right). \tag{E.10}$$

Then we obtain

$$\begin{aligned}\textbf{(con. term)} = {} & e^{i\pi\hat{c}\frac{z^2}{\tau}} \frac{1}{N} \sum_{b\in\mathbf{Z}_N} \sum_{m'\in\mathbf{Z}_{2NK}} e^{-2\pi i\frac{(s-K)b}{N}} e^{-2\pi i\frac{am'}{N}}\, \Theta_{m',NK}\left(\tau,\frac{2z}{N}\right) \times \\ & \times \frac{-i}{\sqrt{2NK}} \int_{\mathbf{R}+i0} dp'\, q^{\frac{p'^2}{2}} \frac{1}{1-e^{-2\pi\left(\frac{p'}{\sqrt{2NK}}+i\frac{m'-2Kb}{2NK}\right)}} \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}. \end{aligned} \tag{E.11}$$

Moreover, using the identities

$$\begin{aligned} \frac{1}{1-e^{-2\pi\left(\frac{p'}{\sqrt{2NK}}+i\frac{m'-2Kb}{2NK}\right)}} &= \sum_{\alpha=0}^{N-1} \frac{e^{-2\pi\alpha\left(\frac{p'}{\sqrt{2NK}}+i\frac{m'-2Kb}{2NK}\right)}}{1-e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)}}, \\ \frac{1}{N}\sum_{b\in\mathbf{Z}_N} e^{-2\pi i\frac{b}{N}(s-K-\alpha)} &= \delta^{(N)}_{\alpha,s-K}, \end{aligned} \tag{E.12}$$

we find

$$\begin{aligned}\textbf{(con. term)} &= -e^{i\pi\hat{c}\frac{z^2}{\tau}} \frac{i}{\sqrt{2NK}} \sum_{m'\in\mathbf{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}}\, \Theta_{m',NK}\left(\tau,\frac{2z}{N}\right) \times \\ &\quad \times \int_{\mathbf{R}+i0} dp'\, q^{\frac{p'^2}{2}} \frac{e^{-2\pi\frac{s-K}{\sqrt{2NK}}p'}}{1-e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)}} \frac{i\theta_1(\tau,z)}{\eta(\tau)^3} \\ &= -e^{i\pi\hat{c}\frac{z^2}{\tau}} \frac{i}{\sqrt{2NK}} \sum_{m'\in\mathbf{Z}_{2NK}} e^{-2\pi i\frac{mm'}{2NK}} \times \\ &\quad \times \int_{\mathbf{R}+i0} dp' \frac{e^{-2\pi\frac{s-K}{\sqrt{2NK}}p'}}{1-e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)}} \chi_{\mathbf{c}}{}^{(\widetilde{\mathrm{R}})}(p',m';\tau,z). \end{aligned} \tag{E.13}$$

In this way we have reproduced the modular transformation formula (E.1).

To complete our proof we have to also work with the cases of $s < K$ and $N + K \leq s$. First of all, $s = N + K$ is reduced to $s = K$ by the charge conjugation relation (C.7). Next we consider the case of $s > N + K$. We set

$$s - K = s_0 - K + Nj, \qquad 0 \leq s_0 - K \leq N, \qquad j \in \mathbb{Z}_{>0}, \tag{E.14}$$

and

$$m = s - K + 2Kr, \qquad m_0 = s_0 - K + 2Kr, \qquad (\text{i.e. } m = m_0 + Nj). \tag{E.15}$$

Making use of the identity

$$\frac{\left(yq^{N\left(n+\frac{r}{N}\right)}\right)^{\frac{s_0-K}{N}+j}}{1-yq^{N\left(n+\frac{r}{N}\right)}}=\frac{\left(yq^{N\left(n+\frac{r}{N}\right)}\right)^{\frac{s_0-K}{N}}}{1-yq^{N\left(n+\frac{r}{N}\right)}}-\sum_{a=0}^{j-1}\left(yq^{N\left(n+\frac{r}{N}\right)}\right)^{\frac{s_0-K}{N}+a}, \tag{E.16}$$

we can show

$$\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s,m;\tau,z)=\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s_0,m_0;\tau,z)-\sum_{a=0}^{j-1}\chi_{\mathbf{c}}{}^{(\widetilde{\mathrm{R}})}\left(i\frac{s_0-K+Na}{\sqrt{2NK}},m_0+Na;\tau,z\right). \tag{E.17}$$

Since $K\le s_0\le N+K$, the first term $\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s_0,m_0;\tau,z)$ has been already shown to obey the modular transformation formula (E.1), and the modular transformation of second term is evaluated as

$$\begin{aligned}&\chi_{\mathbf{c}}{}^{(\widetilde{\mathrm{R}})}\left(i\frac{s_0-K+Na}{\sqrt{2NK}},m_0+Na;-\frac{1}{\tau},\frac{z}{\tau}\right)= \\ &=-e^{i\pi\hat{c}\frac{z^2}{\tau}}\frac{i}{\sqrt{2NK}}\sum_{m'\in\mathbf{Z}_{2NK}}e^{-2\pi i\frac{(m_0+Na)m'}{2NK}}\int_{-\infty}^{\infty}dp'\,e^{-2\pi\frac{s_0-K+Na}{\sqrt{2NK}}p'}\chi_{\mathbf{c}}{}^{(\widetilde{\mathrm{R}})}(p',m';\tau,z)\,.\end{aligned} \tag{E.18}$$

We also remark

$$\frac{e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)\frac{s_0-K}{N}}}{1-e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)}}-\sum_{a=0}^{j-1}e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)\left(\frac{s_0-K}{N}+a\right)}=\frac{e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)\left(\frac{s_0-K}{N}+j\right)}}{1-e^{-2\pi\left(\sqrt{\frac{N}{2K}}p'+i\frac{m'}{2K}\right)}}, \tag{E.19}$$

$$e^{2\pi i\frac{(s-K)(s'-K)-mm'}{2NK}}=e^{2\pi i\frac{(s_0-K)(s'-K)-m_0m'}{2NK}}. \tag{E.20}$$

Combining these identities with the character relation (E.17), we can readily confirm that $\chi_{\mathbf{d}}{}^{(\widetilde{\mathrm{R}})}(s,m;\tau,z)$ with $s>N+K$ still obeys the modular transformation formula (E.1).

The cases of $s<K$ can be reduced to $s>N+K$ by using again the charge conjugation relation (C.7). Therefore, the formula (E.1) has been derived from (E.6) in all the cases.

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